EXPLORING TEACHERS' CATEGORIZATIONS FOR AND CONCEPTIONS OF COMBINATORIAL PROBLEMS

Nicholas H. Wasserman
Southern Methodist University
nwasserman@smu.edu

While counting is simple enough, counting problems span the spectrum of difficulty. Although mathematicians have succinct categories for differing problem types, students struggle to model solving problems and to identify related problem structures. In a graduate course, K-12 mathematics teachers (n=7) were introduced to combinatorial problems and then given a set of problems to solve and categorize. Results from this study specify ways that mathematics teachers who are also novice combinatorialists identified similarities between problems; two particularly difficult problems reveal poignant conceptions and explanatory categorizations.

With increased emphasis on probability in K-12 mathematics education (e.g., Common Core State Standards, 2010), knowledge of combinatorial thinking is becoming increasingly necessary for both students and teachers (e.g., counting the cardinality of sets and sample spaces). Permutations and combinations, while frequently included in the curriculum, are often tangential topics in the scope of mathematics learning and only superficially discussed. Kapur (1970) noted potential benefits for integrating combinatorics into the K-12 curriculum, which include making conjectures, thinking systematically, one-to-one mappings, and many applications in physics, biology, and computer science. The rapid pace and content coverage required for state exams may be one source of blame for the current disintegration; however, another probable reason is a lack of knowledge or comfort with combinatorics on the part of teachers. Counting problems can be very challenging, and, while expert mathematicians have succinct categories for differing problem types, the process for learning to think combinatorially may not be so neatly packaged. This paper looks at categorizations for and conceptions of different combinatorics problems made by middle and secondary mathematics teachers; interesting findings and implications for the learning and teaching of combinatorial problems are discussed.

Literature

While counting is simple enough, counting problems span the spectrum of difficulty. Even authors of combinatorics textbooks weigh in on the difficulties encountered and insights required in such problems (e.g., Tucker, 2002). One issue in learning combinatorics is finding appropriate ways to model specific problems. Batanero, Navarro-Pelayo, & Godino (1997) discuss three different implicit models – selections, distributions, and partitions – for combinatorial problems;

furthermore, each model may result in different solutions based on other structures within in the problem. Identifying common structures, beyond modeling, within otherwise dissimilar problems also serves as a barrier to the learning and teaching of combinatorics (e.g., English, 1991) – despite the fact that expert mathematicians have identified nice categories for different counting problems according to the common 2x2 matrix of: with and without repetition, and ordered and unordered selection (see Table 1). While many problems may require more than one of these four approaches, even the basic distinctions between these problem types may not be fully understood by novice learners, particularly given the variety of modeling techniques. In fact, the connections (or lack thereof) made by novices as they solve counting problems can provide insight into common conceptions and misconceptions faced during the learning process.

Table 1
Selecting k objects from n distinct objects

	Ordered (permutations)	**Unordered** (combinations)
Without repetition	Arrangements $\frac{n!}{(n-k)!} = n \times (n-1) \times (n-2) \times .. \times (n-k+1)$	Subsets $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
With repetition	Sequences $n^k = \underbrace{n \times n \times n \times .. \times n}_{k}$	Multisubsets $\left(\binom{n}{k}\right) = \binom{k+n-1}{n-1}$

Adapted from: Benjamin, A.T. (2009, p. 10)

Identifying ways to apply knowledge from previously learned problems to another context is generally known as *transfer*. The roots of transfer extend back to behaviorism, where the idea was viewed as fundamental to the learning process. More recently, however, alternatives and adaptions to the traditional view of transfer have been articulated; in particular, Lobato (2003) characterizes *actor-oriented transfer* (AOT). AOT shifts the perspective regarding transfer from an expert's view to a learner's vantage point, which results in paying particular attention to the ways that novices draw on their knowledge to solve new problems. Lockwood (2011) argues that AOT is a particularly poignant perspective for investigating combinatorial learning because the subject depends strongly on "establishing structural relationships between problems" (p. 309). Given the importance of combinatorial thinking for and the current emphasis on understanding probability and statistics, efforts using AOT to investigate how such thinking develops, for students and teachers, are warranted. Specifically, this paper addresses the following question: How do middle and secondary mathematics teachers who are also novice combinatorialists categorize and conceptualize different combinatorial problems?

Methodology

As a starting point for investigating how middle and secondary mathematics teachers that are novice combinatorialists categorize and conceptualize various problem types, two focus groups (e.g., Berg & Lune, 2012) were conducted (N=3 and N=4). The focus groups were conducted in conjunction with a graduate mathematics education course; all seven participants in the focus groups were practicing middle and secondary teachers with less than 6 years teaching experience and were enrolled in the course. The focus groups were preceded by a brief introduction to combinatorics problems in the course. While the middle and secondary teachers in the course had various mathematical backgrounds, none of the focus group participants had completed a course in combinatorics or discrete mathematics, making them novice combinatorialists.

The brief introduction (~90 minutes) in the course consisted of two parts: 1) overt instruction on the addition principle; the multiplication principle; factorial notation; and dividing out extraneous solutions when order is irrelevant, including the $\binom{n}{k}$ notation; and 2) approaches and solutions to six combinatorics problems, which were selected as being relatively common examples of the four types of problems from textbooks and other literature. The six problems (see Table 2) were presented to students as a way to expose them to various strategies for solving combinatorial problems; no structural characteristics of problems (e.g., order matters, repetition allowed) were mentioned and no connections between "types" of problems were discussed.

Table 2
Description of Combinatorics Problems presented to participants with solutions

Name	**Description**	**Type & Solution**
Handshake	If 10 people are at a party and everyone shakes hands with everyone else, how many total handshakes are given?	Subset $\binom{10}{2}$
Password	A password has to be 8 characters long and can use any of the 26 letters or the 10 digits (not case sensitive). How many different passwords are there?	Sequence 36^8
Hot Dogs	Hot dogs come in 3 varieties: Regular, Chili, Super. How many different ways are there to purchase 6 hot dogs?	Multisubset $\left(\binom{3}{6}\right) = \binom{6+2}{2}$
Voting	Two candidates are running for a club election. In the end, candidate A gets 4 votes and candidate B gets 5 votes. The moderator of the club, however, reads each vote out loud in order. How many different ways could he read out the votes?	Subset $\binom{9}{4} = \binom{9}{5}$
States	How many different "words" can you make with the letters (nonsense words count) in TEXAS? How about in MISSISSIPPI?	Arrangement $5! \quad \frac{11!}{4!4!2!}$
Vowel	You are creating 5 letter words that CAN repeat letters. How many words are there that have at least one vowel?	Sequence $26^5 \quad 21^5$

After instruction in the course, the study participants (N=7) were randomly assigned to one of two focus groups (~120 minutes each), during which they worked together on an assortment of 12 combinatorial problems, ranging in type and complexity. Through the lens of AOT, participants were asked to: "Answer each of the problems and organize them into 'groups' of problems that have similar methods for solving. For each group of problems, provide a brief description of how and why the problems in that group are similar." The twelve problems, along with the six original problems discussed in class, were printed on note cards to facilitate participants' groupings. While focus group participants worked on the problems and discussed ideas with one another, the researcher took field notes about important comments or connections made by participants (i.e., occurrences of AOT), at times asking questions to uncover their thinking. Participants' mathematical work and their final groupings/descriptions were collected for the study. For space purposes, only some of the problems are described in detail as they come up in the discussion and analysis; however, all problems are listed in Table 5 in the Appendix for reference.

Findings

The categorizations and descriptions created by two focus groups of middle and secondary mathematics teachers provide some information regarding AOT in the learning of counting problems. Generally, participants were able to make and describe the structural connections about *permutations* (Arrangements and Sequences), where order matters, much easier than *combinations* (Subsets and Multisubsets), where order does not matter. With the exception of the Vowel problem, which involved subtracting two sequences, groups were able to identify 100% of the possible Arrangement and Sequence problems (Table 3). The groups, however, also placed extra problems in these categories (reasons are discussed later). In addition, the focus groups were able to portray the structural similarities between these two problem types precisely: both descriptions explicitly state the appropriate characteristics related to order and repetition.

Table 3

The groups' categories and descriptions for permutation problems

Type	**Problems**	**Group 1**		**Group 2**	
Arrangements (Ordered, without repetition)	Problems States Netflix Plane Routes	Problems States (Texas) Netflix Plane Routes	Description -Each thing can only be in one place at a time (no repeats within set) -Order matters	Problems States Netflix Plane Routes *M/F Committees*	Description -The order of choices matters. -Choices cannot be repeated.
Sequences (Ordered, with repetition)	Problems Password MC Exams1 Gift Cards 4-letter words Vowel	Problems Password MC Exams1 Gift Cards 4-letter words [in Cases] *Marbles*	Description -Things being distributed to different positions -Order matters -One element can be repeated (people getting more than one card)	Problems Password MC Exams1 Gift Cards 4-letter words [missing]	Description -Certain amount of spaces and each space has the same number of options. -Options can be repeated.

The two focus groups had much more difficulty categorizing and describing *combination* problems (Subsets and Multisubsets). While an apparent difference exists between the two groups in their ability to identify common structures between Subset problems (Group 2 successfully accounted for 4 of the 5), both groups had particular difficulty with Multisubset problems (Table 4). Group 1 was unable to solve any of these problem types: indeed, the Marbles problem was incorrectly solved as a Sequence and the Hot Dogs problem, which was solved in class, was connected to simple Subset problems (the group did not appreciate the unique characteristics of the original problem, which was then translated through a stars and bars model to a simple Subset problem). Group 2 solved the Skittles problem and was able to connect it to the Hot Dogs problem; however, rather than focusing on the similar characteristics of these problems, their description for this category was procedural (see Table 4), which demonstrates less sophisticated expertise (e.g., Schoenfeld & Hermann, 1982) and indicates that learners may have difficulty identifying common structures within combination problems.

Table 4

The group's categories and descriptions for combination problems

Type	Problems	Group 1		Group 2	
Subsets (Unordered, without repetition)	Problems Handshakes Supreme Court Voting MC Exams2 M/F Committees	Problems [missing] Supreme Court Voting [did not do] [did not do] *Hot Dogs*	Description -How many different positions each element can occupy (9 choose 6)	Problems Handshakes Supreme Court Voting MC Exams2 [in Arrangement]	Description -Take the groups and choose a certain number. Using the "group" choose "number" gets rid of the duplicates. The duplicates exist because order does not matter.
Multisubsets (Unordered, with repetition)	Problems Hot Dogs Summed Digits Skittles Marbles Pizza Toppings	Problems [in Subset] [did not do] [did not do] [in Sequence] [did not do]		Problems Hot Dogs [did not do] Skittles [did not do] [missing]	Description -Broke into groups to account for no duplicates. The barriers separated into groups. Barriers made choosing easy and allowed for choosing all of one type.

Lastly, Group 1 had an additional category; the two problems identified in this category, Vowel and MC Exams2, were characterized as a "Way to choose a minimum number of outcomes (1 vowel, 2 vowels, etc.). Elements have different characteristics, different groupings. Must look at characteristics as a subset of population." In other words, they viewed problems as similar if they were best solved by splitting into "cases," which, for both problems, was an accurate statement and approach. This gives an indication that, at times, participants made connections according to similar processes instead of structurally similar characteristics.

Discussion

While many findings could be explored in more detail, we will focus on the insight gained from two particularly difficult problems: Gift Cards and Pizza Toppings. These results from the focus groups potentially shed some light on the learning and teaching of combinatorics problems.

The Preferred Vantage Point

The Gift Cards problem (i.e., How many ways can you distribute a $1, $2, $5, $10, and $20 gift card to 8 friends?) is a Sequence problem, which, overall, students were able to solve. However, as an individual case, this problem caused surprising difficulty. (Solving the problem also caused over-generalization to other problems with repetition: "like the Gift Card problem," Group 1's reason for including the Marbles problem as a Sequence.) Both groups began by drawing eight slots, one for each person. Their attempts to distribute the five gift cards to these eight people included, among others, $\binom{8}{5}$ (but then "a person could get more than one gift card") and 5^8 (but then “the last person would not have five choices”). Trying to count which person receives which gift card(s) causes modeling difficulties: each person could have anywhere from 0 to 5 gift cards, and sequential models (i.e., eight slots) make the result for subsequent persons dependent on previous ones. To solve it from this perspective would require accounting for each of the seven distinct integer partitions of 5, and then distributing the gift cards according to these possible partitions, which becomes quite complex. It was not until the participants shifted from the *perspective of the people*, who are receiving gift cards, to the *perspective of the gift cards*, which are being distributed, that progress was made. This shift requires accounting for five gift cards (not eight people): each gift card can be given to any one of eight people (i.e., 8^5). However, taking the perspective of a gift card, as opposed to a person, is less natural – I could care less about to whom *every gift card* gets distributed than to which gift cards *I* am going to receive. The exceptional difficulty encountered by initially modeling the problem from the people’s perspective may provide some implications for the teaching and learning of counting problems. In particular, given that counting problems can frequently be modeled from both of two different perspectives, there seems to be a potential limitation or misconception associated with *the preferred vantage point*, characterized by novices having difficulty modeling combinatorics problems from the less natural (but combinatorially easier) perspective.

Another Approach To Multisubset Problems

The Pizza Toppings problem (i.e., How many ways are there to make a pizza with 2 toppings, if the choices were pepperoni, olives, sausage, ham, mushrooms, and anchovies (double toppings allowed)?), technically, is a Multisubset problem, $\left(\binom{6}{2}\right)=\binom{2+5}{5}$, with repetition and unordered selection. However, participants split it into the sum of two Subset problems: two different toppings, $\binom{6}{2}$, and two identical toppings, $\binom{6}{1}$. In fact, this solution is insightful because it mirrors their (unsuccessful) attempts at solving other Multisubset problems, such as the Summed Digits problem (i.e., How many numbers between 1 and 10,000 have the sum of their digits equal to 9?). Participants tried to simplify by first selecting one, two, three, or four place values (Thousands, Hundreds, Tens, and Ones) on which to distribute the sum of 9 (the leftover place values being assigned a zero). For example, if you only choose one place value, $\binom{4}{1}$, then there is only one way to produce a sum of 9 for each (i.e., 9000, 0900, 0090, 0009); however, if you choose three place values, $\binom{4}{3}$, then the sum of 9 can be accomplished by accounting for the partitions of 9 that use three values (i.e., (7, 1, 1), (6, 2, 1), (5, 2, 2), (5, 3, 1), (4, 3, 2), (4, 4, 1), (3, 3, 3)) and ordering those partitions to account for repeated values. This model for solving the Summed Digits problem was the participants' natural approach, though quite complex. In fact, after my own investigation, all Multisubset problems can be solved using this approach – although the numerous computations quickly become burdensome. The general solution to a Multisubset problem, $\left(\binom{n}{k}\right)=\binom{k+n-1}{n-1}$, can be proved to be equivalent to: $\sum_{i=0}^{k-1}\binom{n}{k-i}\binom{k-1}{i}$. The first term in this sum accounts for the various cases, e.g., $\binom{6}{2},\binom{6}{1}$ in the pizza problem or $\binom{4}{4},\binom{4}{3},\binom{4}{2},\binom{4}{1}$ in the Summed Digits problem, and the second term quantifies the different ways each of those cases can occur within a given problem.

Conclusion

The findings from this study indicate that learners are able to structurally connect and characteristically conceptualize *permutation* problems (ordered selections) with more ease than *combination* problems (unordered selections). Likely, the sequential modeling of problems, which frequently is useful and naturally lends itself to ordered selections, may contribute to the difficulty accounting for unordered selections. The *preferred vantage point* for modeling may also limit novices' abilities to solve counting problems; teachers should be aware of both perspectives and a learner's tendency toward the more natural (or preferred) perspective. Multisubset problems were found to be the most difficult to solve; indeed, unordered selection with repetition requires a fundamental reconception about the problem. For example, for the Summed Digits problem to emphasize unordered selection with repetition would give peculiar solutions like HTTTHOThOH (i.e., 1,332). Participants' work on the Pizza Toppings problem also indicates a different way to approach solving Multisubset problems, potentially more aligned with novices' development. While the counting computations in this method become increasingly prohibitive, the process could be used as a transitional stage that provides students with a natural way to connect to the problem and increasingly moves toward more efficient algorithms. Overall, the perspective from middle and secondary teachers within this study presents some ideas about the learning and teaching of counting problems that, while not claimed with absolute certainty, are of interest and merit further exploration and investigation.

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APPENDIX

Table 5

Description of Combinatorics Problems in Focus Group for participants to solve

Name	Description	Type & Solution
MC Exams1	An exam contains 15 multiple-choice questions, each with 4 choices. How many possible ways of answering these 15 questions are there?	Sequence 4^{15}
Plane Routes	A plane starts in New York City and will travel to 7 different cities before it returns. How many different ways can the plane do this?	Arrangement 7!
Supreme Court Decisions	In how many different ways can the nine members of the Supreme Court reach a six-to-three decision?	Subset $\binom{9}{6}=\binom{9}{3}$
Summed Digits	How many numbers between 1 and 10,000 have the sum of their digits equal to 9?	Multisubset $\left(\binom{4}{9}\right)=\binom{9+3}{3}$
Gift Cards	How many ways can you distribute a \$1, \$2, \$5, \$10, and \$20 gift card to 8 friends?	Sequence 8^5
Skittles	16 skittles go into the small Halloween skittle bags. There are 5 colors to choose from in each bag – Red, Green, Yellow, Orange, and Purple. How many different possible bags of skittle are there?	Multisubset $\left(\binom{5}{16}\right)=\binom{16+4}{4}$
The M/F Committees Problem	There are 7 women and 4 men in a club. How many different 4-person committees have at least two women?	Subset $\binom{7}{2}\binom{4}{2}+\binom{7}{3}\binom{4}{1}+\binom{7}{4}$
Netflix	You have 24 different movies on your Netflix account. In how many different ways could you order them?	Arrangement 24!
MC Exams2	An exam contains 15 multiple-choice questions, each with 4 choices. In how many of the possible ways to answer the exam are at least 10 correct?	Subset $\sum_{k=10}^{15}\binom{15}{k}$
4-letter words	Suppose you make a 4-letter "word" (nonsense words count) from the letters A, B, C, D, and E, where you can repeat letters. How many different "words" are possible?	Sequence 5^4
Marbles	How many ways are there to distribute 25 indistinguishable marbles into 7 different containers?	Multisubset $\left(\binom{7}{25}\right)=\binom{25+6}{6}$
Pizza Toppings	How many ways are there to make a pizza with 2 toppings, if the choices are pepperoni, olives, sausage, ham, mushrooms, and anchovies (double toppings allowed)?	Multisubset $\left(\binom{6}{2}\right)=\binom{2+5}{5}$