

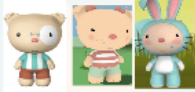
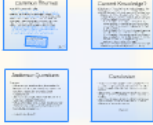




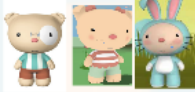
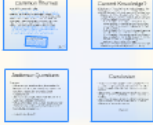
The impact of teachers' knowledge of group theory on teaching practices

Nick Wasserman, Teachers College, Columbia University
Julianna Stockton, Sacred Heart University

	a	b	c	d
a	Background 			
b		Research Questions Does knowledge of new content about algebraic structures, groups in particular, have an impact on teachers' understanding of group theory? If so, in what ways? 		
c			Findings Luis, Franco, Chris 	
d				Discussion 

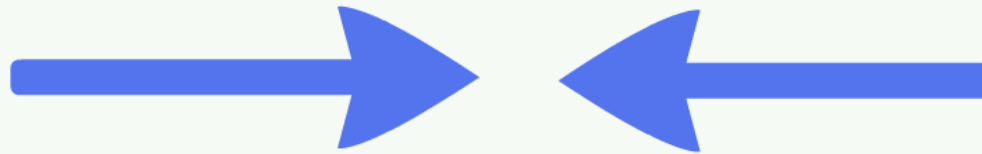
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Teachers' Content Knowledge

It's important... right?



Strong content means: less time to understand material; more time to consider instructional strategies; tend to be more flexible and confident; and have better substance to their teaching (Brown & Borko, 1992)

Profound Understand of Fundamental Mathematics (Ma, 1999)

Yet taking more mathematics courses does not necessarily improve instruction or student performance (Monk, 1994; Darling-Hammond, 2000)

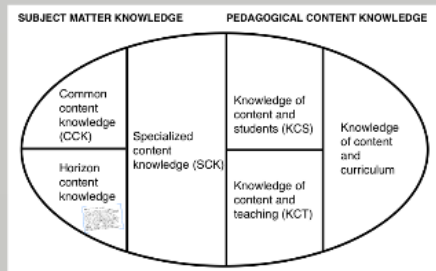
Easy to find deviations - strong teachers who lack formal content instruction, or those strong in content who are ineffective teachers (Davis & Brown, 2009)

Domains of Knowledge

Mathematical Knowledge for Teaching

e.g., Ball, Thames, Phelps, 2008

TYPES of content knowledge in teaching



Knowledge for Algebra Teaching

e.g., McRory, et al. (2012)

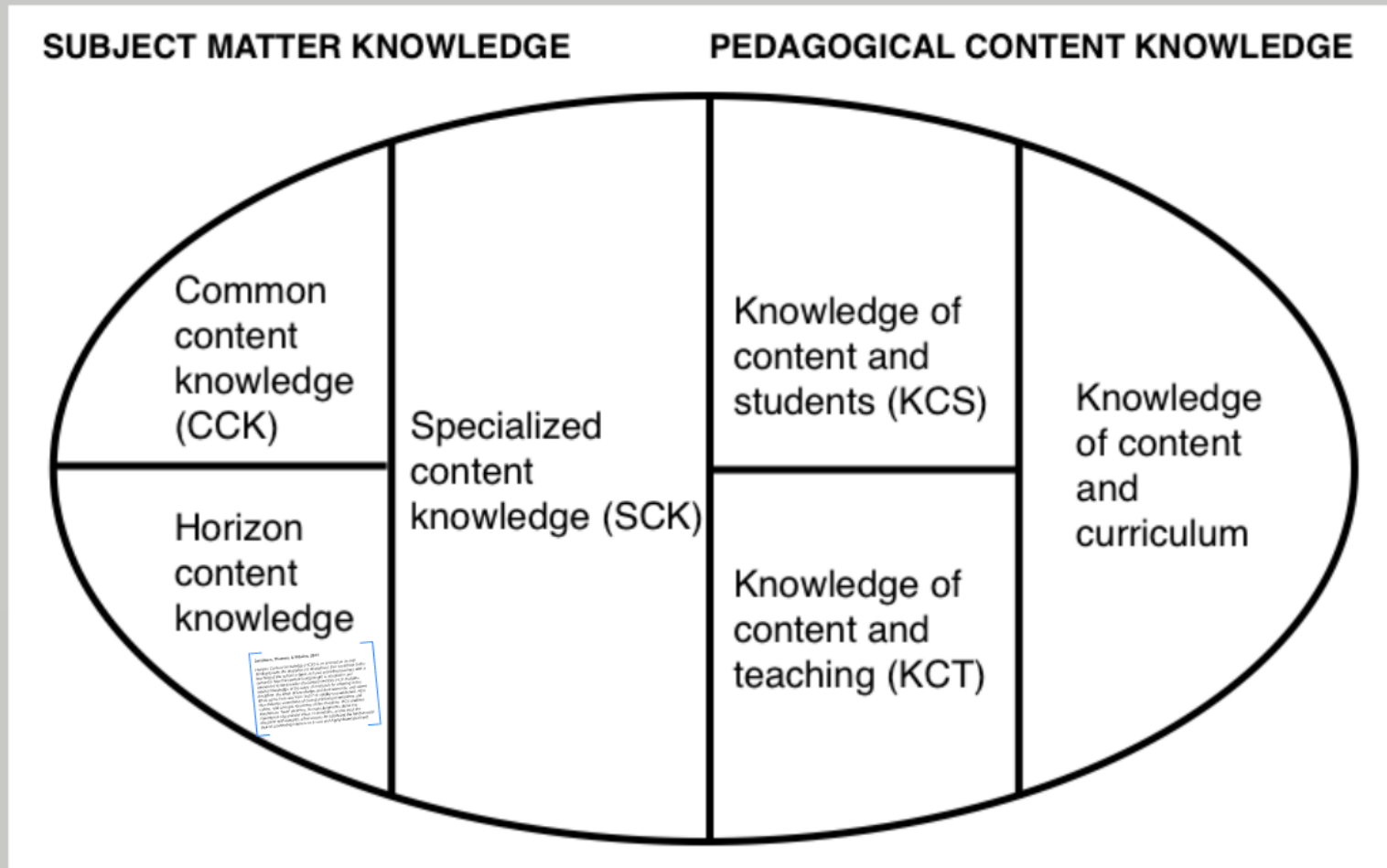
TYPES of content knowledge in teaching

Knowledge of School Algebra **Knowledge of Advanced Mathematics** **Mathematics-for-Teaching Knowledge**

Mathematical Knowledge for Teaching

e.g., Ball, Thames, Phelps, 2008

TYPES of content knowledge in teaching



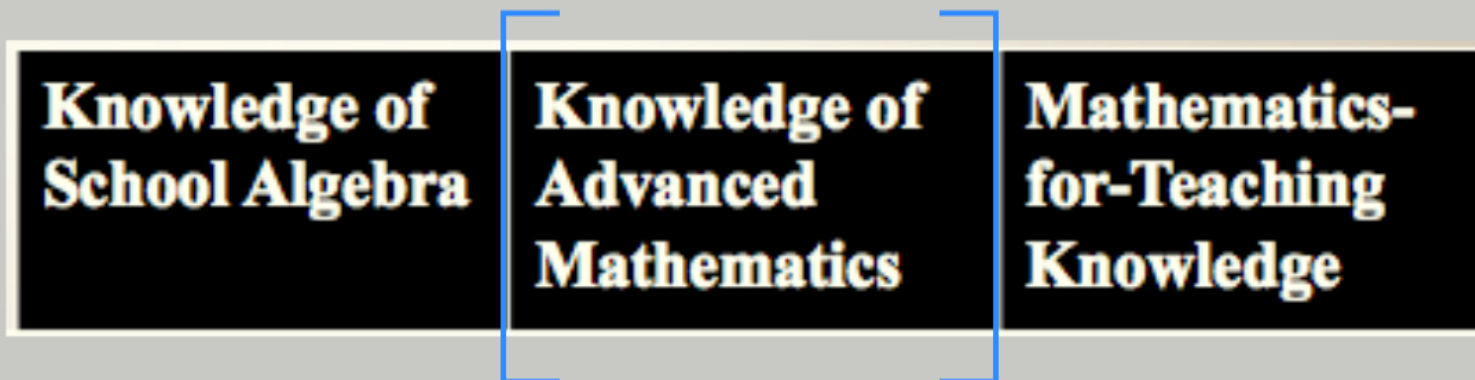
Jakobsen, Thames, & Ribeiro, 2013

Horizon Content Knowledge (HCK) is an orientation to and familiarity with the discipline (or disciplines) that contribute to the teaching of the school subject at hand, providing teachers with a sense for how the content being taught is situated in and connected to the broader disciplinary territory. HCK includes explicit knowledge of the ways of and tools for knowing in the discipline, the kinds of knowledge and their warrants, and where ideas come from and how “truth” or validity is established. HCK also includes awareness of core disciplinary orientations and values, and of major structures of the discipline. HCK enables teachers to “hear” students, to make judgments about the importance of particular ideas or questions, and to treat the discipline with integrity, all resources for balancing the fundamental task of connecting learners to a vast and highly developed field.

Knowledge for Algebra Teaching

e.g., McRory, et al. (2012)

TYPES of content knowledge in teaching



Evidence of Knowledge

Knowledge for Algebra Teaching

McRory, et al. (2012)

How content knowledge is USED in teaching

Trimming

Removing complexity while maintaining integrity

Bridging

Making connections across topics, assignments, representations, and domains

Decompressing

Unpacking complexity in ways that make it comprehensible

Knowledge Quartet

e.g., Rowland, Huckstep, & Thwaites, 2005

How content knowledge is EVIDENT in teaching

Foundation

Theoretical understanding of pedagogy
Assessment of student
Identifying pupil errors
Overt display of subject knowledge
Use of mathematical knowledge
Adherence to textbook
Consistency in procedures

Transformation

Teacher demonstration
Use of instructional materials
Choice of representations
Choice of examples

Connection

Making connections between procedures
Making connections between concepts
Acquisition of conceptual
Decision about sequencing
Recognition of conceptual appropriateness

Contingency

Responding to students ideas
Deviation from lesson agenda
Teacher insight
Responding to the (un)availability of tools and resources

Knowledge for Algebra Teaching

McRory, et al. (2012)

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Adherence to textbook
Concentration on procedures

Transformation

Teacher demonstration
Use of instructional materials
Choice of representations
Choice of examples

Connection

Making connections between procedures
Making connections between concepts
Anticipation of complexity
Decisions about sequencing
Recognition of conceptual appropriateness

Contingency

Responding to students ideas
Deviation from lesson agenda
Teacher insight
Responding to the (un)availability of tools and resources

Conceptualizing Algebra (including Early Algebra)

Algebra is much more than techniques to solve for "x"

Kaput (2008) delineates algebra as:

- The study of functions and joint variation
- The study of structures and systems abstracted from computations and relations
- The application of modeling languages in and out of mathematics

Research Questions

Does knowledge of more advanced abstract algebraic structures, groups in particular, have an impact on (early) algebra teaching practices? If so, in what ways?

Practice-based theories of teacher knowledge

Content Knowledge

Horizon Content Knowledge

Knowledge of Group Theory

Content standard:
Advanced topics
Abstract Algebra
Mathematical Proficiency



impacts?

Analyzed CCSS-M (2010) to identify specific K-12 (in this case K-5) content that might be informed by knowledge of abstract algebraic structures, particularly groups

Reflecti

A group $(G, *)$ is defined as a set of elements, G , together with a binary operation, $*$, that fulfills four axioms:

1. Closure: $\forall a, b \in G, a * b \in G$
2. Associative property: $\forall a, b, c \in G, (a * b) * c = a * (b * c)$
3. Identity element: $\exists e \in G, s.t. \forall a \in G, e * a = a * e = a$
4. Inverse elements: $\forall a \in G, \exists a^{-1} \in G, s.t., a * a^{-1} = a^{-1} * a = e$

Audience Question:
What would you do to solve this equation: $x + 5 = 12$?

Refer up to the work.
Q1: What would you do to solve this equation?
Q2: How might knowledge of advanced mathematics, particularly algebraic structures, be useful to you when teaching students in secondary schools to solve this equation?

Audience Question:

What would you do to solve this equation: $x + 5 = 12$?

Follow-up (to discuss):

Q1. What assumptions must you make in the process?

Q2. How might knowledge of advanced mathematics, particularly algebraic structure, be useful for and inform teaching practices? In elementary/middle/early algebra? secondary?

impacts?

Analyzed CCSS-M (2010) to identify specific K-12 (in this case K-5) content that might be informed by knowledge of abstract algebraic structures, particularly groups

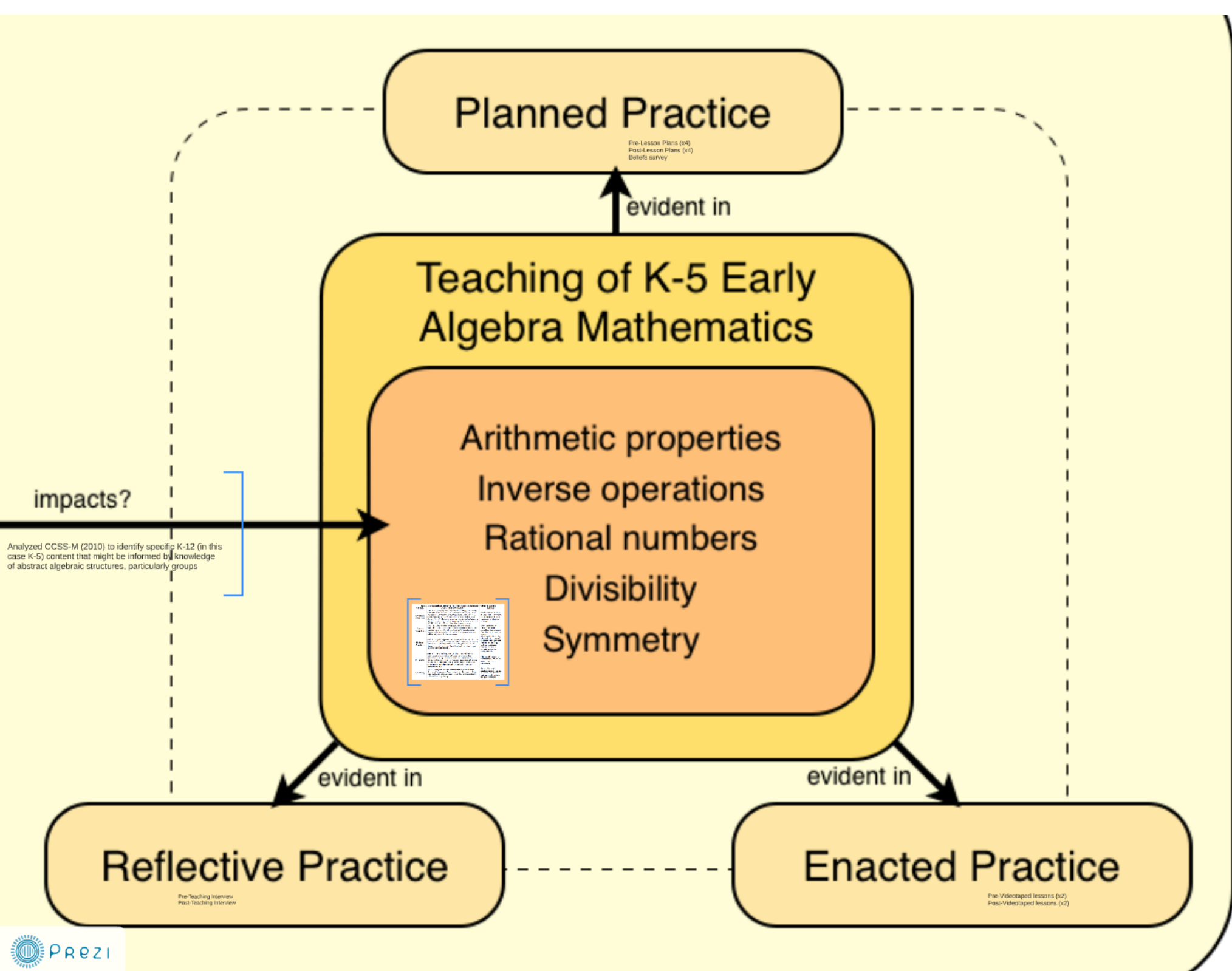
Teaching of K-5 Early Algebra Mathematics

Arithmetic properties
Inverse operations
Rational numbers
Divisibility
Symmetry



Table 1: Conceptualized categories for group theory impacting early algebra teaching

Category	CCSS-M Standard (example)	Rationale
Arithmetic Properties	3.OA.B.5. Apply properties of operations as strategies to multiply and divide. <i>Examples: If $6 \times 4 = 24$ is known, then $4 \times 6 = 24$ is also known. (Commutative property of multiplication.) $3 \times 5 \times 2$ can be found by $3 \times 5 = 15$, then $15 \times 2 = 30$, or by $5 \times 2 = 10$, then $3 \times 10 = 30$. (Associative property of multiplication.) Knowing that $8 \times 5 = 40$ and $8 \times 2 = 16$, one can find 8×7 as $8 \times (5 + 2) = (8 \times 5) + (8 \times 2) = 40 + 16 = 56$. (Distributive property.)</i>	Algebraic groups are based on a collection of <i>arithmetic properties</i> , which form the foundations for algebraic reasoning.
Inverse Operations	5.NBT.B.7. Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.	Inverse operations are informed by inverse elements and the extension of number sets in search of <i>closure</i> .
Rational Numbers	4.NF.A.1. Explain why a fraction a/b is equivalent to a fraction $(n \times a)/(n \times b)$ by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.	The <i>multiplicative identity</i> , 1, for example, is important for constructing equivalent fractions, and reducing fractions. In reducing fractions, the inverse element produces the identity element.
Divisibility	4.OA.A.3. Solve multistep word problems posed with whole numbers and having whole-number answers using the four operations, including problems in which remainders must be interpreted. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.	Solving problems where remainders (<i>additive modulo n</i>) can be operated on independently.
Symmetry	4.G.A.3. Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry.	The set of isometric transformations is complete (or <i>closed</i>) with reflections, rotations, transformations, and glide reflections.



Field Practic

Pre-Lesson Plans (x4)
Post-Lesson Plans (x4)
Beliefs survey

ed Practic

Pre-Videotaped lessons (x2)
Post-Videotaped lessons (x2)



Reflective P

Pre-Teaching Interview
Post-Teaching Interview

Knowledge

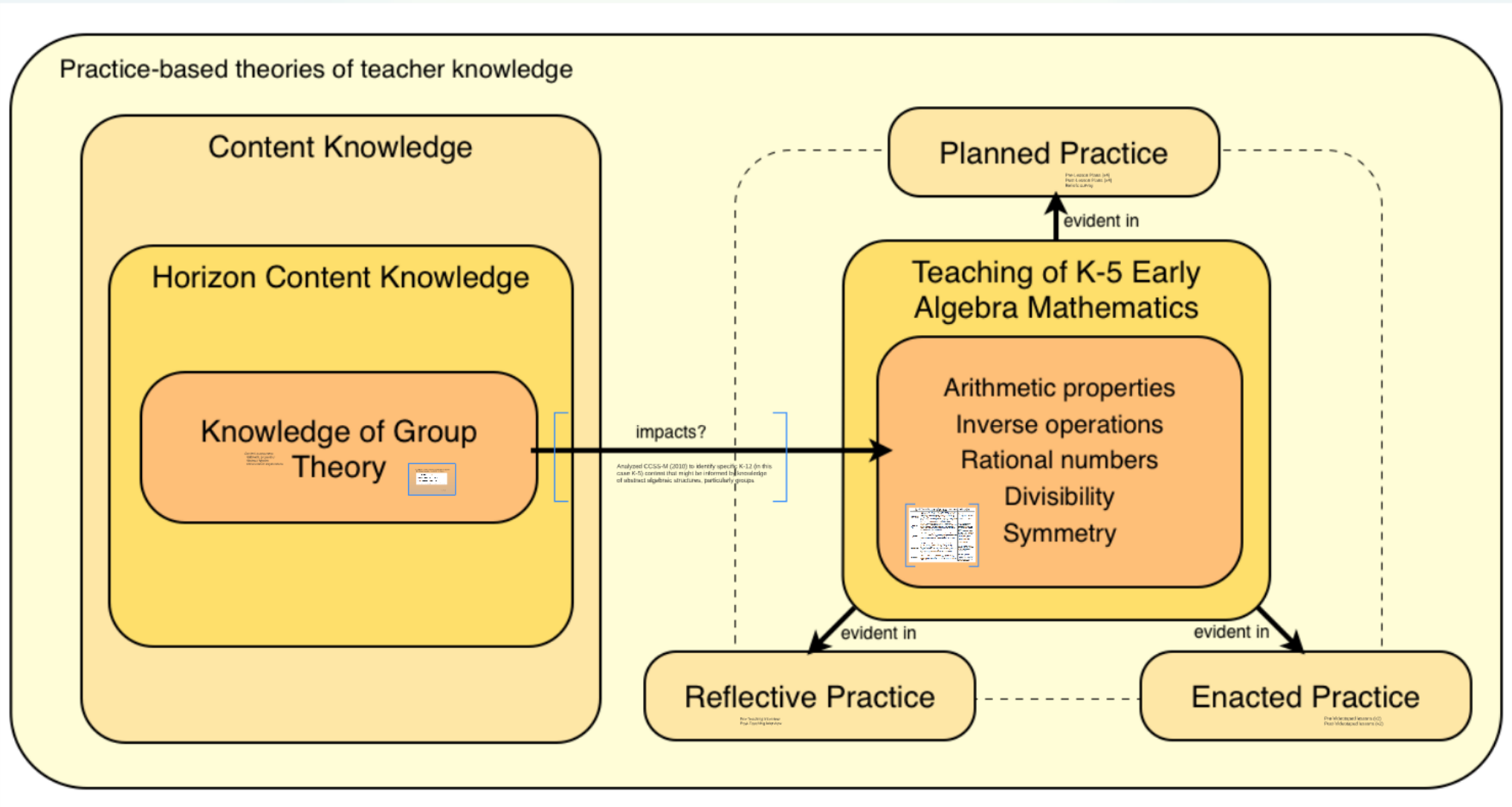
Content assessment:

Arithmetic properties

Abstract Algebra

Mathematical Explanations

Th



From a larger s
report on 3 ele

From a larger study, we will report on 3 elementary teachers



Lewis



- Teaches 5th grade at an elementary school in a large urban school district
- Predominantly (88%) Hispanic student body
- 5 years teaching experience
- Currently pursuing M.A. in bilingual education and a state-level certification as a Master Mathematics teacher
- Previously taken Calc I and II courses

Francis



- Teaches 4th & 5th grade, and math coach for elementary in a large suburban school district
- Diverse mix of African American, Caucasian, and Hispanic students
- Veteran teacher of 31 years; recently awarded peer-nominated district-level award
- Earned both bachelors and masters degree in elementary education
- Previously taken Calc I and Prob/Stat courses

Cindy



- Teaches 5th grade in a middle school (5-8), part of a nationally reputable network of charter schools
- Almost entirely African American and Hispanic student body
- 9 years of teaching experience
- Undergraduate psychology degree, with minor in human development
- Never taken Calc course; one undergrad Stats

Findings

Lewis



Francis



Cindy



Lewis

Pre-teaching:

- Focus on describing rules and tricks without opportunity for sense-making



- Development of concepts with vocabulary lack rationale and importance



Post-teaching:

- Highlights rules and properties as meaningful exercises in sense-making



- Attention to development of concepts with vocabulary, particularly arithmetic properties



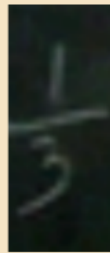
(pre-teaching): focus on using rules and tricks

On equivalent fractions:

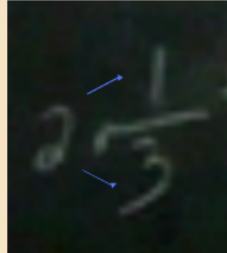
- "We cancel the like numbers"

$$\frac{2}{20} = \frac{2 \times 1}{2 \times 10}$$

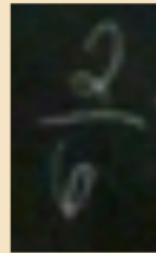
- "What you do to the top you do to the bottom"



Handwritten fraction $\frac{1}{3}$ on a chalkboard.



Handwritten fraction $\frac{2}{6}$ on a chalkboard. Two blue arrows point to the numerator '2' and the denominator '6'.



Handwritten fraction $\frac{1}{3}$ on a chalkboard.

Evidence of:

- Inaccurate trimming (phrases inaccurate, addition?)
- Concentration on procedures; lack of connection between procedures

Evidence of:

- Inaccurate *trimming* (phrases inaccurate, addition?)
- Concentration on procedures; lack of connection between procedures

(pre-teaching): development of concepts with vocabulary lacks rationale

Teacher: What do you call fractions that represent the same quantity?
Students (all): Equivalent fractions
Teacher: That's correct.
[moves on]

Key Vocabulary:

1. Fractions
2. Numerator
3. Denominator
4. Multiple
5. Addition
6. Subtraction
7. LCD
8. Reduce
9. Mixed numbers

On arithmetic properties...

Teacher: Lets pretend that the number 5 has a denominator (writes $5/$). If you have to choose a denominator that does not change the number 5, what would it be?

Students: [Silent]

Teacher: Like if you have to put a denominator right here, but does not change the number 5, what number would it be?

Students: [Silent]

Teacher: What number divided by 5 equals 5? Which one? One.
[writes $5/1$].

Evidence of:
Use of mathematical terminology in some cases
Lack of awareness of purpose

Evidence of:

Use of mathematical terminology in some cases

Lack of awareness of purpose

(post-teaching): rules and properties as meaningful exercises

Purposeful Instruction

WARM UP (5 min): Two students are working on the following multiplication problem: 37×18 . One student arranges the problem like this:

$$\begin{array}{r} 37 \\ \times 18 \\ \hline \end{array}$$

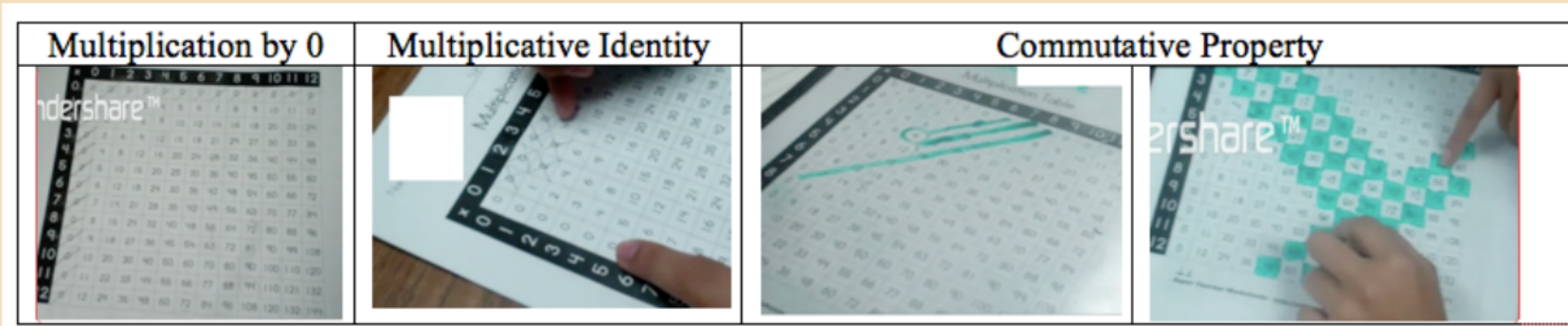
and the other student does this: $\begin{array}{r} 18 \\ \times 37 \\ \hline \end{array}$

Who do you think is correct? Are the products of these two problems different? Solve and share with your pair.

[perhaps unfortunate that product is 666]

(post-teaching): rules and properties as meaningful exercises

Looking for patterns in a 12x12 multiplication table



Ideas connected to early algebra structure, and mathematics more generally, should be taught as a reasoning and sense-making activity [in reference to 12x12 multiplication table lesson]:

“I think the fact of finding patterns and developing that skill in students at that early age makes a difference in how they see the mathematics.”

(from interview)

- Evidence of:
- Bridging
 - Decompressing
 - Awareness of purpose
 - Use of mathematical terminology
 - Choice of examples
 - Making connections between procedures
 - Making connections between concepts
 - Responding to students ideas

Evidence of:

- *Bridging*
- *Decompressing*
- *Awareness of purpose*
- *Use of mathematical terminology*
- *Choice of examples*
- *Making connections between procedures*
- *Making connections between concepts*
- *Responding to students ideas*

(post-teaching): attention to development of concept & rationale with vocabulary

Rationale for vocabulary

Students are given a set of color tiles to model math operations and review some of the basic number properties of addition.

Using tiles:

- Using tiles represent $5+6$. Have the students find the total. Then ask, does it make any difference in the result if we switch the numbers to have $6 + 5$? Elaborate further. (*Commutative Property*)
- Using tiles represent $5+6$. Have students find the total. Then, using the same tiles, represent $5 + 5 + 1$. Ask, does it make any difference in the result if I add $5 + 5$ first and ten 1? Provide more examples. (*Associative Property*). Elaborate further.
- Using tiles represent $9 + 0$. Have students find the total. Then ask, what happens to a number, every time you add zero? (*Identity Property*). Elaborate further.

for prediction

"So, if I multiply $2,000,000 \times 0$, what is going to be the product?"

for ease

"While before you might have to know 144 numbers ("facts"), now you only have to know half."

From interviews:

“Well because in the past I was uncomfortable doing the properties. Actually, I felt that they were just some clinical words that they were not actually useful... Then, as I prepared one of the assigned lesson on the multiplication chart, I realized that students would benefit a lot if they understood the arithmetic properties before introducing 3 digit by 2 digit multiplication.”

“Well for me, the arithmetic properties were just a set of useless statements that students did not need to understand. However, when doing the activity with the [group of triangle symmetries], I started to see that the arithmetic properties were more than just unpractical statements. I believe that the fact that we were *actually seeing* what [abstract] properties such as closure and inverse looked like changed my perception of them.”

Francis

Pre-teaching:

- Plans to discuss arithmetic properties, but did not do so



- Incomplete explanations of mathematical ideas

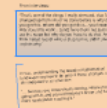


Post-teaching:

- Connected and consistent reinforcement of arithmetic properties



- More thorough explanations, pushing students toward understanding why



(pre-teaching): plans to discuss arithmetic properties, but does not

- On equivalent fractions: recognizes multiplicative identity, though never references in class

together.

Demonstrate finding a common factor and dividing the numerator and denominator to find a equivalent fraction. Also demonstrate multiplying the numerator and the denominator by the same number (equivalent fraction of 1) to also find an equivalent fraction.

- Briefly mentions inverse, though no further clarification

“Now I'm getting you to double check yourself. Because multiplication is the inverse property of division. That means opposite, it's opposite.”

Evidence of:

- Inaccurate trimming
- Concentration on procedures
- Lack of connection between concepts
- Lack of use of mathematical terminology

Evidence of:

- Inaccurate trimming
- Concentration on procedures
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(pre-teaching): incomplete mathematical explanations

Incomplete and somewhat confusing explanation for generating *equivalent fractions*

Proportions

Teacher: [Pausing in the middle of reading a story book.] **Fractions are just another way to divide by. And we have 2 children and 12 cookies (writes fraction $2/12$).** [Draws 12 cookies on board. Circles 6 as one group, and other 6 as other group]. So we have two groups out of 12. How many cookies are each of them going to get?

Students: 6

Teacher: So I could go, and **just one of them is going to get 6, right, so up here we have 1 out of 6 (writes fraction $1/6$). And I put equals $2/12$.** Two children, 12 cookies, and each one is going to get 6. So I can make that an equivalent fraction, that's the same amount of cookies isn't it. Just divided up another way.

Fraction Strips

“Equivalent fractions are the same amount, it’s just broken up differently.”

Note:

Equivalent fractions and equivalent proportions are different: in one, the whole gets smaller, with fewer pieces of the same size; in the other, the whole remains the same, with more pieces of a smaller size

Evidence of

- Potentially confusing choice of representations
- Lack of connections between concepts

Evidence of:

- Potentially confusing *choice of representations*
- Lack of *connections between concepts*

(post-teaching): consistent and connected discussion of arithmetic properties

As her introduction to a lesson on equivalent fractions:

Teacher: Today we're going to talk about something that you know, but you may not know the word for it. Has anybody every heard of the identity property of 0? [Silence] You have your boards there, I want you to write this for me: $6 + 0 = \underline{\quad}$. What does it equal?

Students: 6.

Teacher: Let's try another one. Let's put $1,002+0=\underline{\quad}$. What does it equal?

Students: 1,002.

Teacher: Did you need your boards for that?

Students: No!

Teacher: Why not?

Student (one of many hands up): **Because every number added to 0 will stay the same.**

Teacher: Identity property. Identity is you, isn't it? **Your identity is who you are. 6 is 6 isn't it? We add 0 to it, it's still 6, it still has the same identity.** So it kind of goes along with that.

Teacher: There's a number for multiplication that does that. **Can you think of the number for multiplication that no matter what you multiply the number stays the same?**

...

Teacher: So we have the identity property for addition is 0, and the identity property for multiplication is 1. You've been using it already this year, you just didn't know what it was called. So when you see it today, you'll go, 'I know what that is.'

Evidence of:

- Bridging
- Decompressing
- Awareness of purpose
- Use of mathematical terminology
- Making connections between concepts
- Recognition of conceptual appropriateness

Evidence of:

- *Bridging*
- *Decompressing*
- *Awareness of purpose*
- *Use of mathematical terminology*
- *Making connections between concepts*
- *Recognition of conceptual appropriateness*

(post-teaching): more thorough explanations

Explaining equivalent fractions

Teacher: And you all know that whatever you do to the bottom you do to the top. *I'm going to tell you why.* I ended up with the number $2/2$. What's another way to say that? That's the same as 1 whole. We're multiplying by 1.

Student: It's itself.

Teacher: The identity property of 1. *So when we multiply by 1, we get the same number. So when we take $1/2 \times 2/2$, its the same as $1/2 \times 1$,* its the same number. Equivalent fractions are the exact same number.

Making ideas explicit to students, not assuming they know them:

“Verbalization of the concepts and writing it out [is so important]. I know I was guilty, especially in teaching math. I just kind of did that but went over it fast.”

(from interview)

From interviews:

“That’s one of the things I really stressed, that I’ve changed up from what I’ve done before is *why* do you get this, where did you get that... you know... *why does this work*... [and] have them not just write out the steps but why do you have to do that. And I think I used to just write a step but we didn’t always explain why.”

In fact, understanding the broader mathematical landscape seemed to be part of these changes, as she indicated in an interview:

“... because you know what’s coming, where you’re going with it, and you understand it better and feel more comfortable teaching it.”

Cindy

Pre-teaching:

- Emphasis on rules, tips, and tricks to obtain correct answers

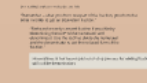


- Consistently uses vocabulary, though not for arithmetic properties

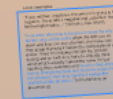
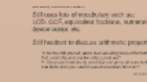


Post-teaching:

- Emphasis on rules, tips, and tricks to obtain correct answers



- Consistently uses vocabulary, though not for arithmetic properties



(pre-teaching): emphasis on rules, tips, and tricks

Frequently students recite rules and "smartcuts"

The rule to remember is:

What you do to the top of the fraction, you must also do to the bottom of the fraction!

Problem Solving

Directions: Do the DR QVOSAC steps to solve the following problem.

1) The staff versus 8th grade basketball game was awesome! Before the game, Mrs.

Banks predicted that the teachers would make 27 of the **2-point shots**. How many points will the teachers have in all if Mrs. Banks predicted correctly?

"...so let's just say I want [students] to look at the hundreds place or round to the hundreds place. They have to know this is zero over here *so underline the place, look to the right, are you more than four? No, I don't grow* (emphasis added). So this just stays the same, so you copy down the left and that's where the left and right came in and the right becomes zero."

(from interview)

Her rule about equivalent fractions gets her into a predicament:

(Student): So $1/2$, multiply $1 \times 3 = 3$ and $2 \times 3 = 6$, so $3/6$. And then I added +1 on the numerator and on the denominator, and I got $4/7$.

(Teacher): Should we add? (Another student): Cause that's not um..[inaudible]... You can't just add a 1... you have to do like this.

Evidence of:
- Inaccurate trimming
- Concentration on procedures; lack of connection between procedures

Evidence of:

- Inaccurate trimming
- Concentration on procedures; lack of connection between procedures

(pre-teaching): inconsistent use of vocabulary

Frequently uses mathematical and non-mathematical vocabulary

What?
<i>Define the key concepts and terms associated with the knowledge-based aim.</i>
Rounding
Estimation
Approximately
Almost
About
Nearest
Closest

Equivalent Fractions

Students will SLANT and monitor comprehension

DR QVOSAC

While Cindy discusses arithmetic properties, somewhat surprising (given tendency toward vocabulary) that she does not use vocabulary in these instances:

Student: 4 times 0

Class (in unison): 0

Teacher: 4

Class: (pause) [inaudible] You're wrong

Teacher: I would like to agree with the fact that I am wrong.

But how many times have I seen people say that 0×4 or $0 \times$ anything is that number?

Student: That's times 1.

Teacher: Oh, okay, that's times 1.

Student: They're thinking of adding 0.

Teacher: Oh, okay. So we need to check our what?

Class (in unison): Work

(post-teaching): emphasis on rules, tips, and tricks

“Remember – what you do to one part of the fraction, you do to the other in order to get an equivalent fraction.”

“Reduce the newly created fraction if possible by determining the GCF of the numerator and denominator. Use the GCF to divide the numerator and the denominator to get the reduced form of the fraction.”

She outlines in her lesson plan an 8-step process for adding fractions with unlike denominators

Evidence of:
- Inaccurate trimming
- Concentration on procedures; lack of connection between procedures

Evidence of:

- Inaccurate *trimming*
- *Concentration on procedures; lack of connection between procedures*

(post-teaching): inconsistent use of vocabulary

Still uses lots of vocabulary such as:
LCD, GCF, equivalent fractions, numerator,
denominator, etc.

Still hesitant to discuss arithmetic properties:

"I: So how did you talk about that canceling out part for them that a red chip and a yellow chip cancel out?"

P: If you owed somebody one dollar and you paid somebody one dollar then you would have what amount left over?"

(from interview)

From interviews:

"If you add two negatives, the answer is going to be negative. If you add a negative and a positive, look for the higher value..." [still rules, tips, tricks]

"A lot more effort has to be given toward the *why* behind, *why all this works* when the kids can sit down and they can pay attention and make sure they grasp that *why* it makes the computational part easier. What I'm noticing with like the STAAR testing and so forth they have to critically think about what's actually happening verse I'm just inputting these numbers and *so when they know how to change decimals and fractions and percents into whatever form they need to it makes the problem a little bit easier.*" [still emphasis on procedures]

Discussion

Common Themes

- Across all three participants, initially:
 - Emphasis on procedures and rules
 - Avoiding or not engaging in discussing arithmetic properties
- While not all participants had equal changes in practice, *this was mediated by content knowledge changes*; Lewis and Francis made significant gains in content knowledge whereas Cindy did not
- Learning about more advanced algebraic structures, groups in particular, impacted early algebra instruction (particularly, arithmetic properties, inverse operations, rational numbers):
 - Purposeful use and development of arithmetic properties
 - Turning instruction and exposition about early structures into meaningful exercises in sense-making
 - Emphasis on bridging ideas and concepts for students

Changes in instruction appear related to knowledge of mathematical language.

- More aware of algebraic structures in elementary setting, helped in grappling with those ideas in abstract context.
- More connected understanding of arithmetic, related to collective aspect of properties in algebraic structures.
- More purposeful with arithmetic structures, related to appreciation of broader disciplinary landscape.

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Content Knowledge?

- While Lewis and Francis both demonstrated changes in their teaching practices and their content knowledge, there were some potential differences in their gains in content knowledge
 - Francis demonstrated knowledge of structure in abstract settings, whereas Lewis was unable in abstract settings to completely generalize these ideas.
 - Potential explanation 1: Knowing the mathematical horizon (to have an impact on practice) at times may take the form of "awareness" as opposed to "mastery" of advanced ideas
 - Potential explanation 2: Unrealistic to expect significant leaps in content changes from short introduction. Francis already knew meaning of arithmetic properties, grew to understand them in abstract settings; Lewis did not know them in either setting initially, grew to understand them in arithmetic settings.
 - The content assessments, however, were limited in further understanding differences (improvements in future)

Audience Questions

Conclusion

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• We only had 3 participants
• The study is limited by one institutional approach
• We are discussing change, though not necessarily claiming that the changes are good or bad

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Audience Questions

Discuss:

- What are the merits and/or drawbacks for inclusion of more advanced mathematics in teacher preparation or professional development?
- Are there any implications from this work in your own settings? Describe.
- What other types of evidence would you look for to identify an impact of horizon content knowledge on teaching?
- General questions?

Conclusion

- Our claim: During both teacher education and professional development, requirements for advanced content or courses do not necessarily need to be *more*, they need to be *more informed*.
- More rigorous investigation is needed; in particular, a more fine-grained analysis of advanced content and the impact on teaching
- Mathematics content preparation should be designed as an *intentional* sequence of mathematics content, not just an *arbitrary* set of mathematics courses.

THANKS!