# Exploring the role of the Mathematical Horizon for Secondary Teachers

## NCTM 2015 Research Symposium

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# Introduction

What goes through your mind when you look at the following statement(s)?

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x$$

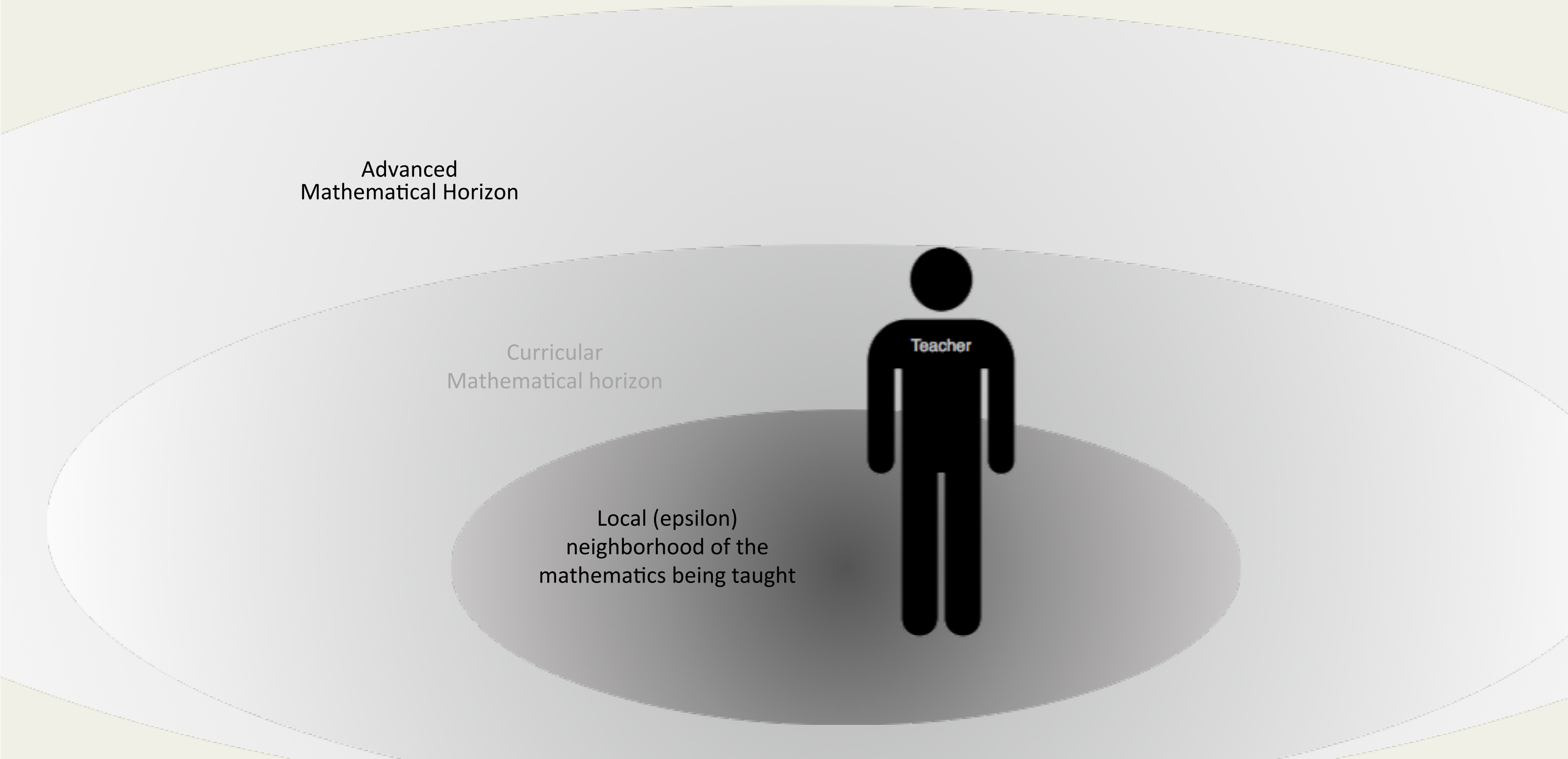
$$s_x = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}}$$

The interior angle sum of a triangle is 180°.

# Teachers' Mathematical Knowledge

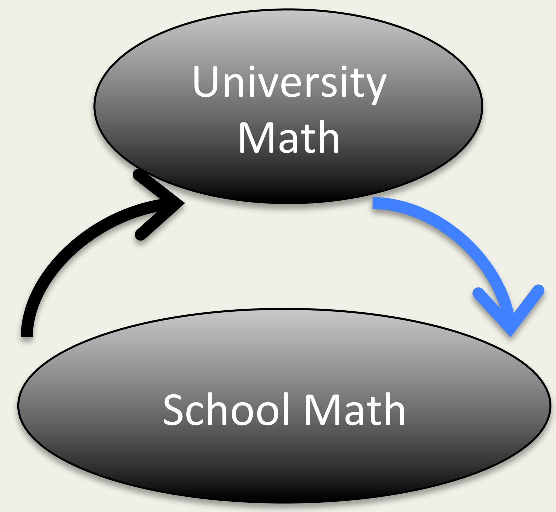
- Strong content for teachers is important but insufficient; teacher knowledge is a complex construct
- Ball et al. (2008) provisionally included Horizon Content Knowledge as part of their MKT framework
  - "Horizon knowledge is an awareness of how mathematical topics are related over the span of mathematics included in the curriculum…It also includes the vision useful in seeing connections to much later mathematical ideas" (p. 403)
- How might knowledge outside the content a teacher teaches be important for their work?

# Mathematical Landscape



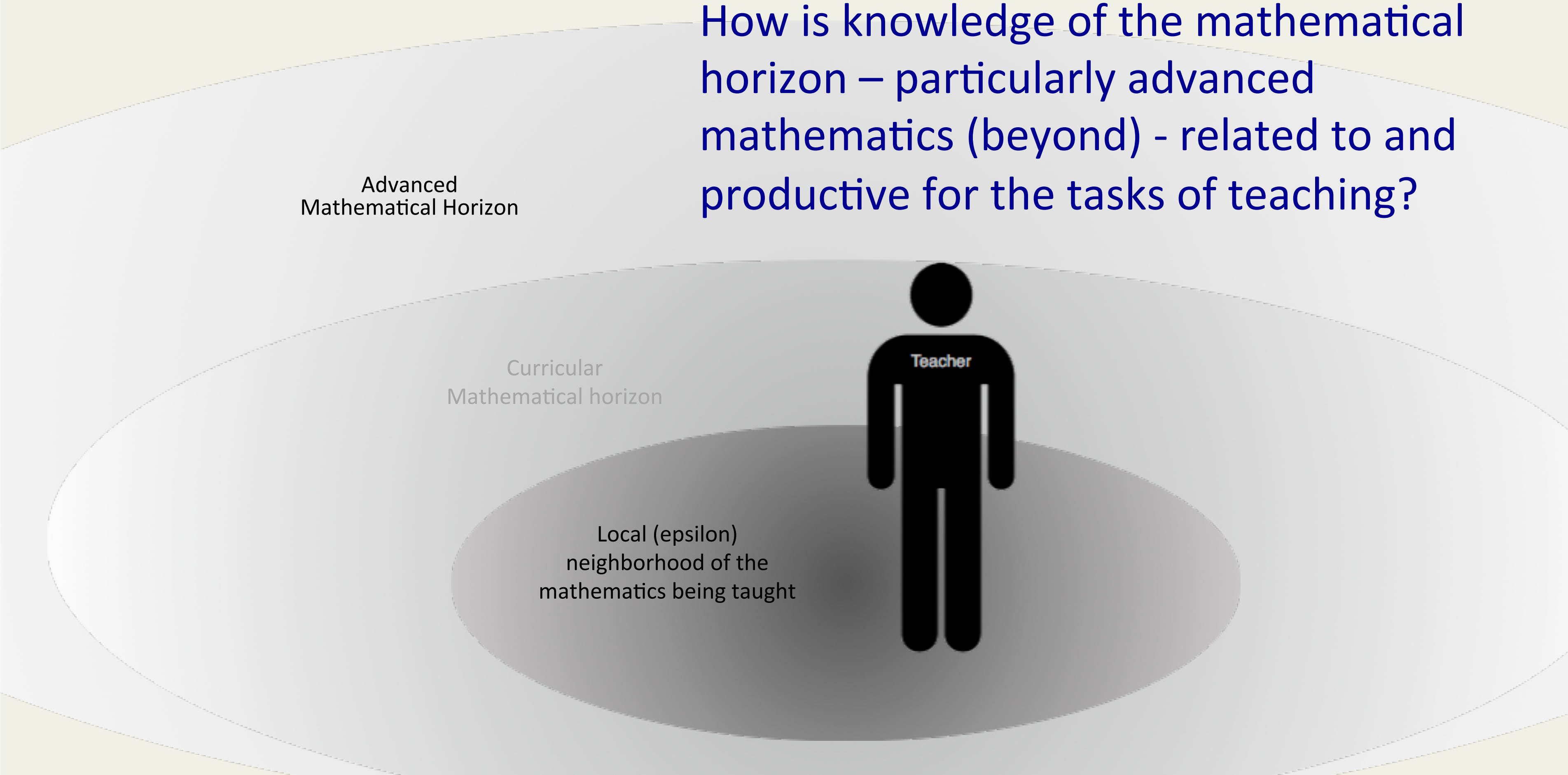
# A “Double Discontinuity”

- Felix Klein (1932) observed what he coined a “double discontinuity” for teachers:
    1. study of university mathematics did not develop from or suggest the school mathematics that students knew
    2. returning back to school mathematics, the university mathematics appeared unrelated to the tasks of teaching.



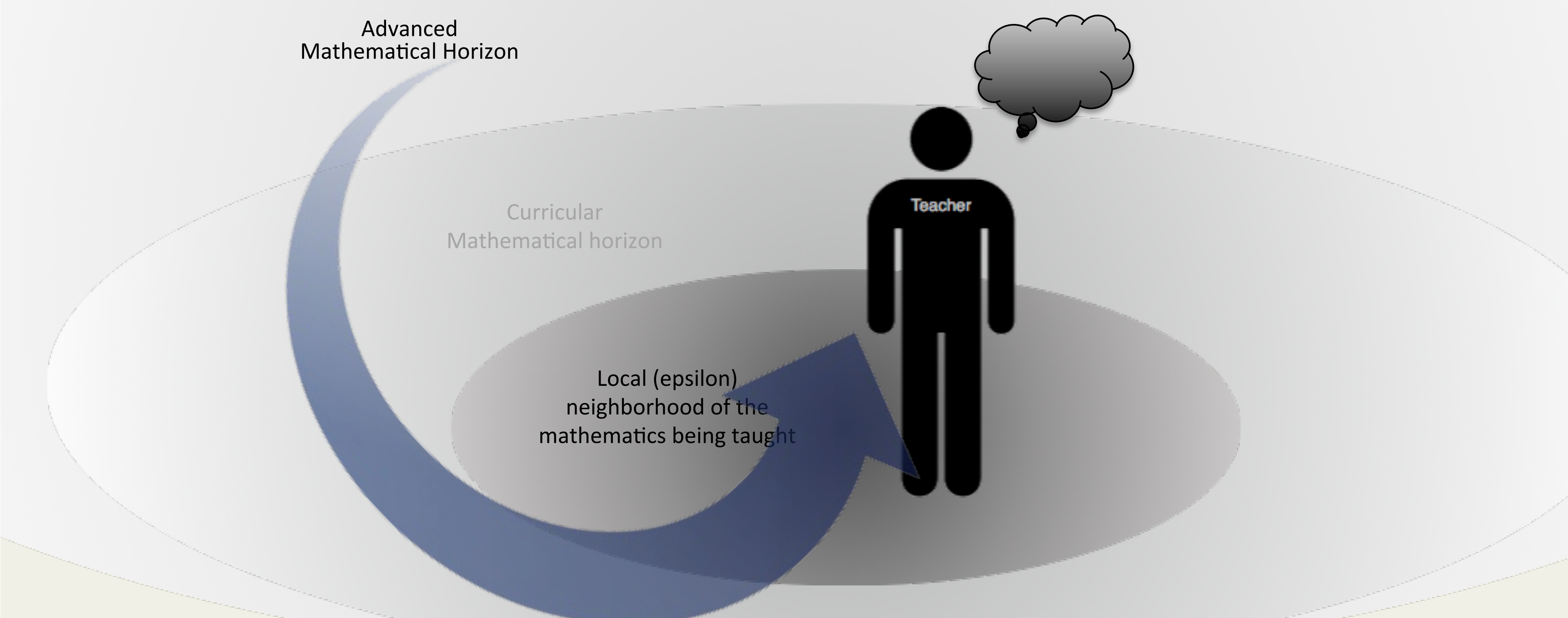
# Mathematical Landscape

How is knowledge of the mathematical horizon – particularly advanced mathematics (beyond) - related to and productive for the tasks of teaching?



# Mathematical Landscape

1. As a key developmental understanding (KDU):
How might more advanced mathematics transform teachers' own understanding and perception of the *local* content they teach in productive ways for their teaching?

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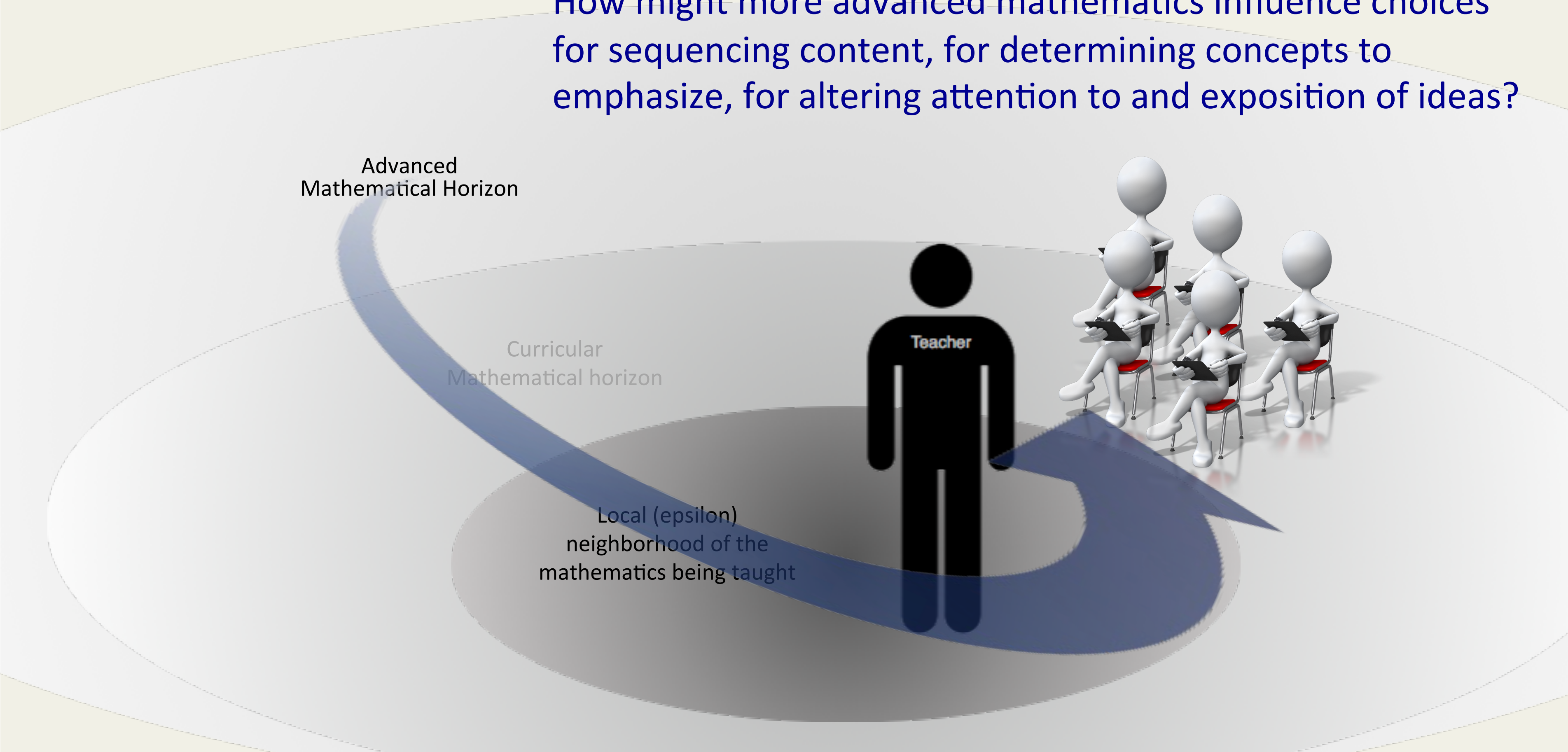
# Mathematical Landscape

2. As more directly influencing practices:
How might more advanced mathematics influence choices for sequencing content, for determining concepts to emphasize, for altering attention to and exposition of ideas?



# Different Studies

- In this session we'll explore studies related to advanced mathematics and secondary teaching
  - Statistics as Unbiased Estimators (Stephanie Casey, Joe Champion, Maryann Huey, Nick Wasserman)
  - Mapping Abstract Algebra for Algebra Teaching (Andrew Sanfratello, Brandie Waid, Nick Wasserman)
  - Real Analysis (ULTRA) (Tim Fukawa-Connelly, Pablo Mejia-Ramos, Matt Villanuea, Keith Weber, Nick Wasserman)
  - Forms of Knowing Advanced Mathematics (Julianna Stockton, Nick Wasserman)

# Participant Discussion

- Are there aspects of more advanced mathematics that might positively influence teachers as they engage in their work and practice? (See handout)
  - In relation to how advanced mathematics might productively change their understandings, awareness, and/or perceptions of the content they teach?
  - In relation to how advanced mathematics might productively influence their teaching practices – in what ways?

# Unbiased Estimators

- Statistics is increasingly important in K-12 mathematics. Statistical thinking, inference in particular (not descriptive), is more difficult to grasp (Casey & Wasserman, 2015)
- Research Question
    - What role does understanding and proving that statistics (standard deviation) are unbiased estimators of population parameters play in teachers' approaches to teaching (standard deviation)?

sample $s_x = \sqrt{\dfrac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}}$ population $\sigma_x = \sqrt{\dfrac{\sum_{i=1}^{n}(x_i - \mu)^2}{n}}$

# Methodology for Study

- Qualitative design, pre/post lesson on SD as an estimator
- Purposeful sampling at 4 sites; 2 preservice (*n* = 8), 2 inservice (*n* = 8)
- Data Analysis
  - Pre-Post test for specialized content knowledge of SD
    - Interpretation of mean, SD, formulas for SD
    - Mean and SD as unbiased estimators
  - Lesson Plans for teaching SD
    - Statistical content (procedures, concepts, n vs. n-1 formula(s)), cognitive demand
  - Post Interviews
    - SD as unbiased estimator
    - Explaining n vs. n-1 to students
    - Would you change the lesson now?
- Interpretation
    - Knowledge of SD as unbiased estimator (Low, Developing, High)
    - Connections between instructional choices and knowledge of SD as estimator

# Teaching Standard Deviation

- Finding 1: In interviews, most teachers expressed dissatisfaction with their original lesson plans on standard deviation.
  - Lesson plans tended to be procedural in nature.

# Finding 1: Pre-Assessment

- Teachers were uncomfortable with components of the standard deviation formula, especially explaining the division by $n$-1 versus $n$.
- Pre-Assessment:
  - 6 people left completely blank
  - 7 people said dividing by *n*-1 corrects for taking a sample.
  - Answers indicate uncertainty of why the correction is necessary.

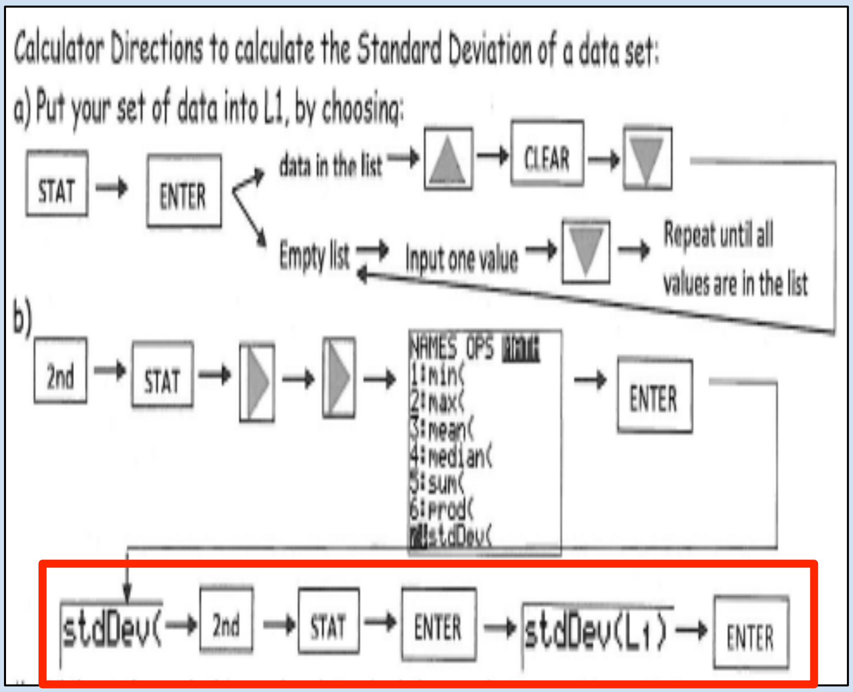
E22: "accommodate for errors in sample data"

B01: "to account for odd points of data, like outliers"

D04: "It's not as accurate as measuring every single one so have to get slightly bigger answer."

# Finding 1: Lesson Plans

- Within Pre-LPs, teachers mostly attended to dividing by *n-1* in a procedural way or avoided the topic.




| [Saying] | [Writing on board] |
|---|---|
| Begin by finding the sample mean, or $\bar{x}$ | |
| Then, we find the difference between each data point and the mean. | |
| Next, we square each of those differences and add them all together. | $S = \sqrt{\frac{\sum_i^n (x_i - \bar{x})^2}{n-1}}$ |
| Then, we divide by one less than the number of data points in the sample. (Remember that *n* is the number of data points in a sample) | |
| And finally we take the square root of all that. | |

# Finding 1: Post-Interviews

- Teachers expressed dissatisfaction with original Pre-LP on Standard Deviation
  - E35: "I would do it in a different way, because like when I wrote this, I hadn't had a lot of familiarity with statistics, [and now] it would be easier to explain the formulas more, because I don't really think I did that. I think I more of just gave them [the formulas] to the students."
  - D05: "When I get to that $n$-1 part I usually just say it is because of the sample. I don't usually have a good reason why…"

# Future Teaching of $n$ vs. $n-1$

- Finding 2: Participants with a better understanding of $s_x$ as an unbiased estimator tended to pursue more student-centered, cognitively demanding approaches for addressing division by $n$ vs $n-1$ with students.

# Finding 2: Post-Interviews

- Differing degrees of student-centered, cognitively demanding approaches to address $n$ vs $n\text{-}1$ with students

**Less student-centered, cognitively demanding approaches:**

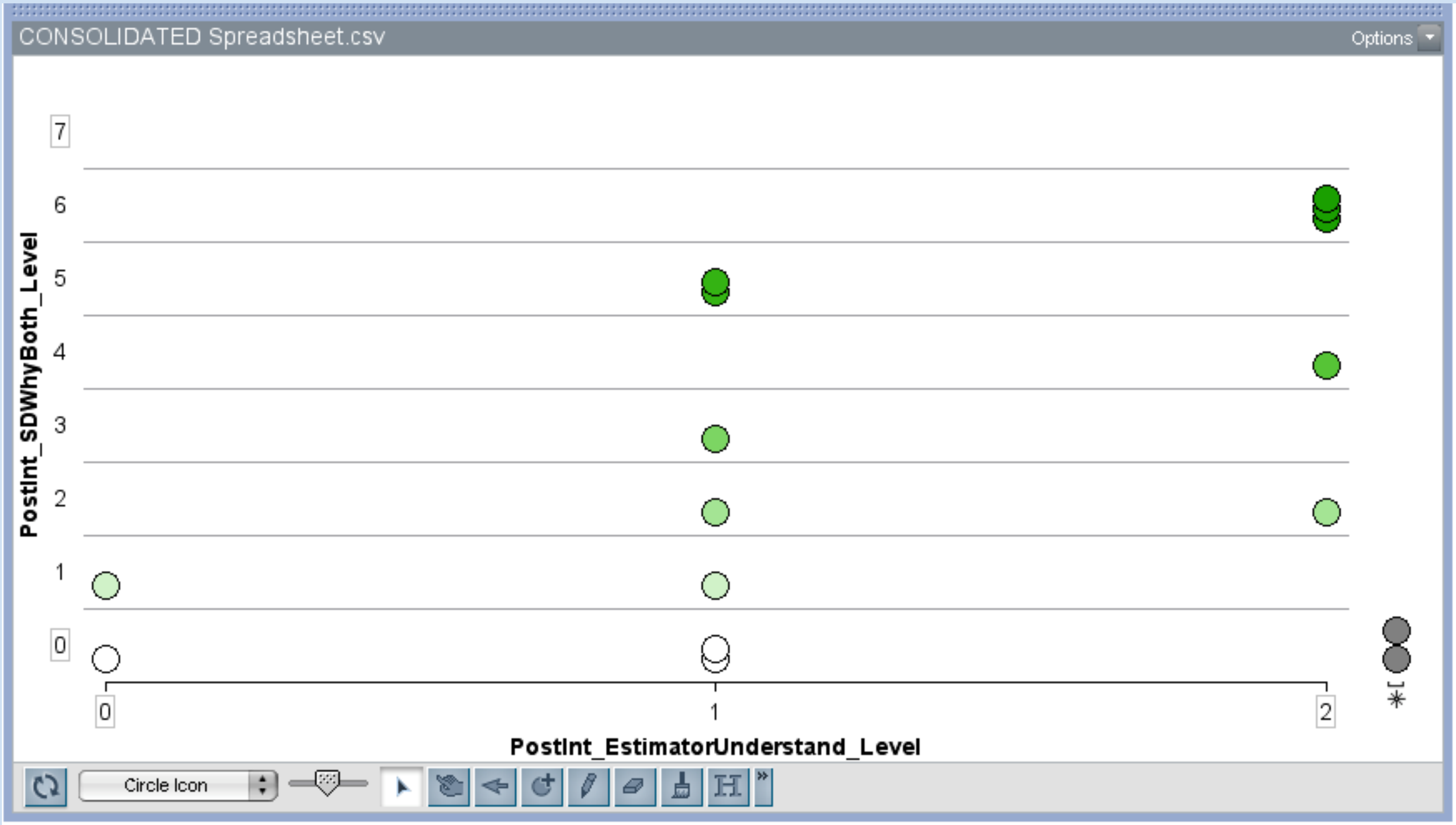
- T02: "I would basically say that we have to subtract one from our population to account for the fact that this isn't our entire data set. So we're bringing our standard deviation up a little bit by subtracting one from our sample size."
- B01: "I would just tell them it divided by $n\text{-}1$ when you're doing a sample because you're not taking the entire population, you're counting for the outliers in the sample."

**More student-centered, cognitively demanding approaches:**

- D05: "I really like [the Fathom] dynamic [task] because the students can engage in them. And you can see how they are convincing…It's easier to say [n-1 is] the degrees of freedom and we will look at it in a later chapter, but that doesn't really answer the question."
- T03: "Ask students to consider the data set {0, 2, 4}, with all possible samples of size *n*=2, calculating the mean of all $s_x$ compared with $\sigma_x$. Repeat this process, but divide by $n$. Try to convince me $n\text{-}1$ is the better divisor."

# Finding 2: Post-Lesson Plans

Increasing degree of student-centered, cognitively demanding activities →




Increasing understanding of estimator →

# Final Thoughts

- Teachers dissatisfaction supports a need to better understand knowledge for teaching standard deviation.
- The process of helping teachers develop an understanding of unbiased estimators was challenging; However, teachers who grasped an inferential notion of estimator were more willing to actively engage students in reasoning & sense-making about formulas.
- While teachers themselves questioned division by $n-1$ instead of $n-2$, etc., and required a formal proof, many found informal approaches, such as Fathom simulations, potentially more useful for teaching.

# Abstract Algebra

- Much of the content of secondary mathematics are examples of more abstract algebraic structures (e.g., the *field* of real numbers with addition and multiplication) – but whether secondary teachers make these connections and whether these are influential for their own teaching is less clear.
- Research question
  - Do teachers, having recently taken a graduate level course in abstract algebra, connect their knowledge of abstract algebra to reshape their understanding and teaching of inverse functions (including function composition)?

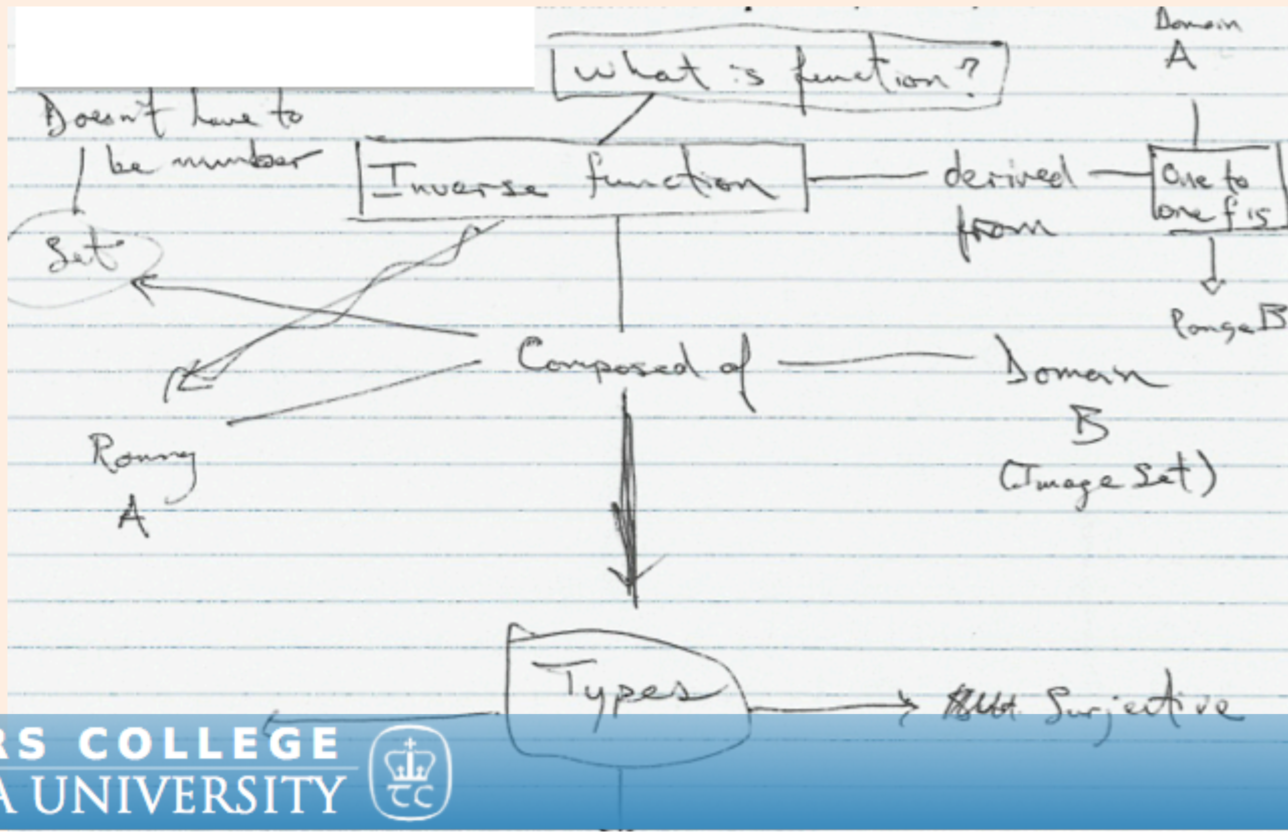
# Methodology for Study

- Participants
  - Selected from a graduate level abstract algebra course
  - Created a stratified, random sample using midterm data
    - 3 high, 2 medium, 2 low; 4 were pre-service, 3 were in-service
- Conducted semi-structured interviews, consisting of 3 parts:
  - Survey of content and teaching knowledge
  - Mathematics connection tasks
  - Impact on teaching
- Coded interviews for data analysis

# Concept Map Question

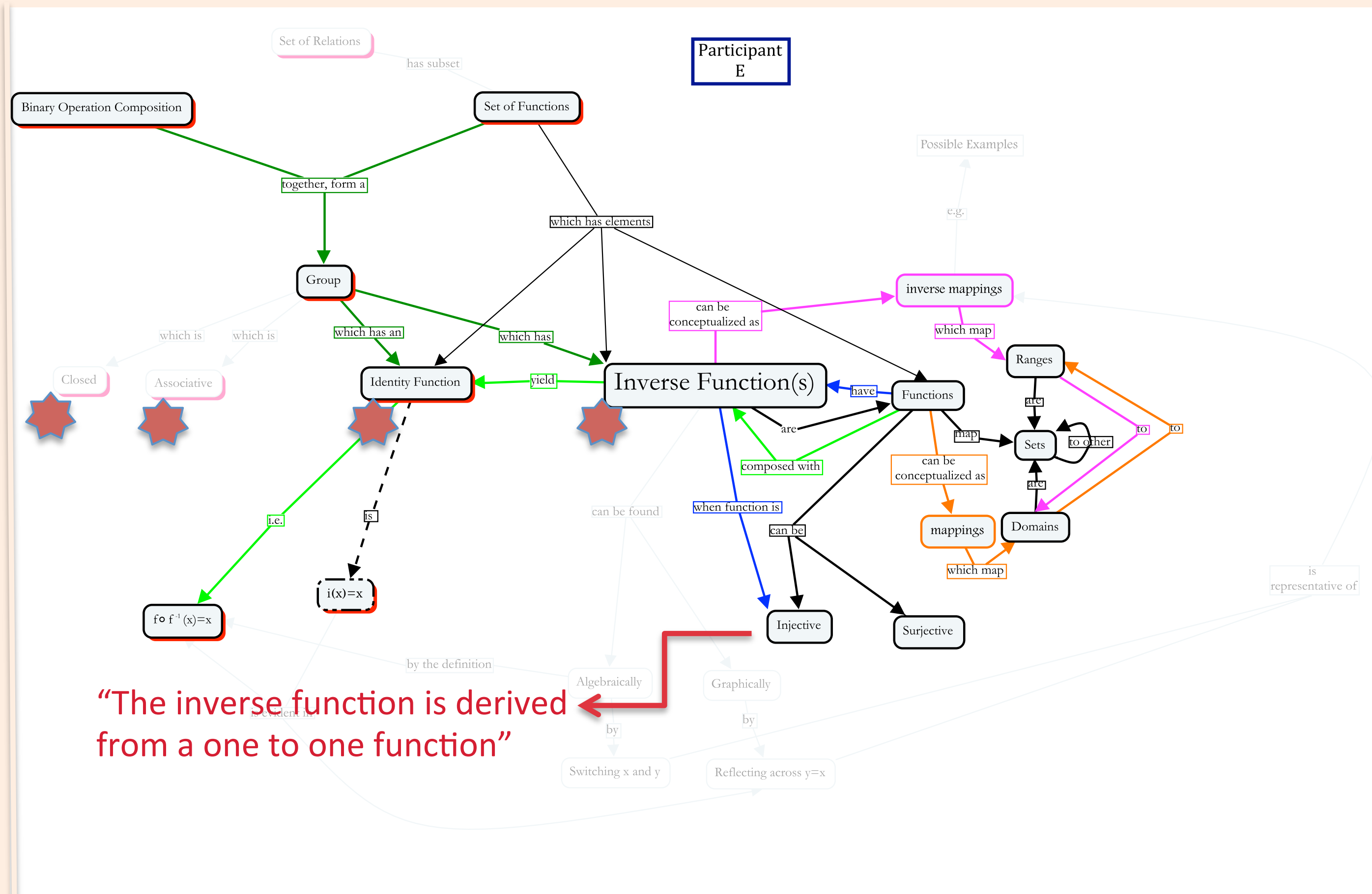
- "Construct a concept map connecting all ideas related to inverse functions. Talk aloud as you make your map, explaining your rationale for each idea."
  - Possible Nodes: Composition, Domain, Identity, Reflection, Set, Injective

Participant E



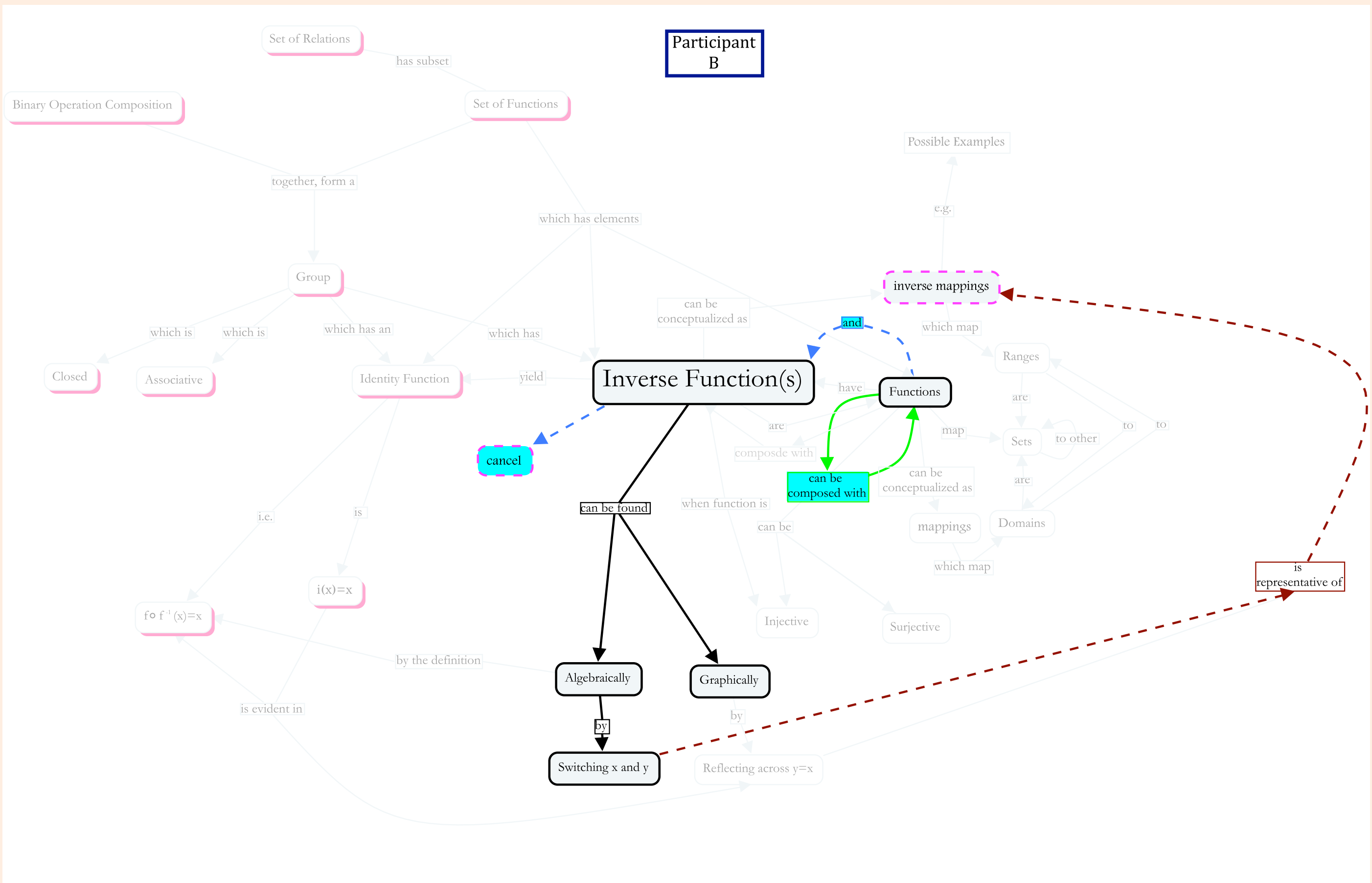
# Concept Map Coding

Participant E
Complex Map (3)



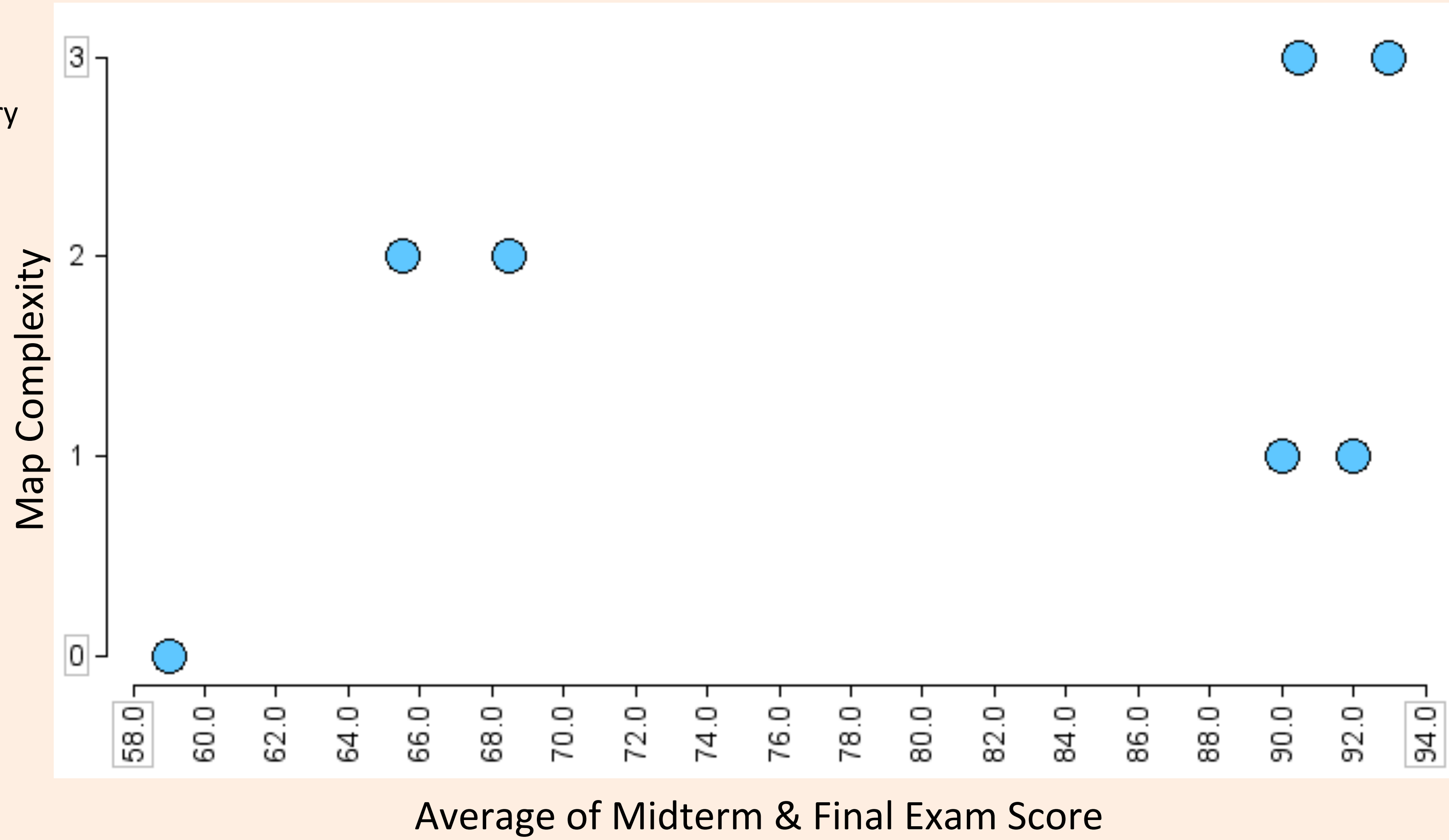
# Concept Map Coding

Participant B
Simple Map (0)



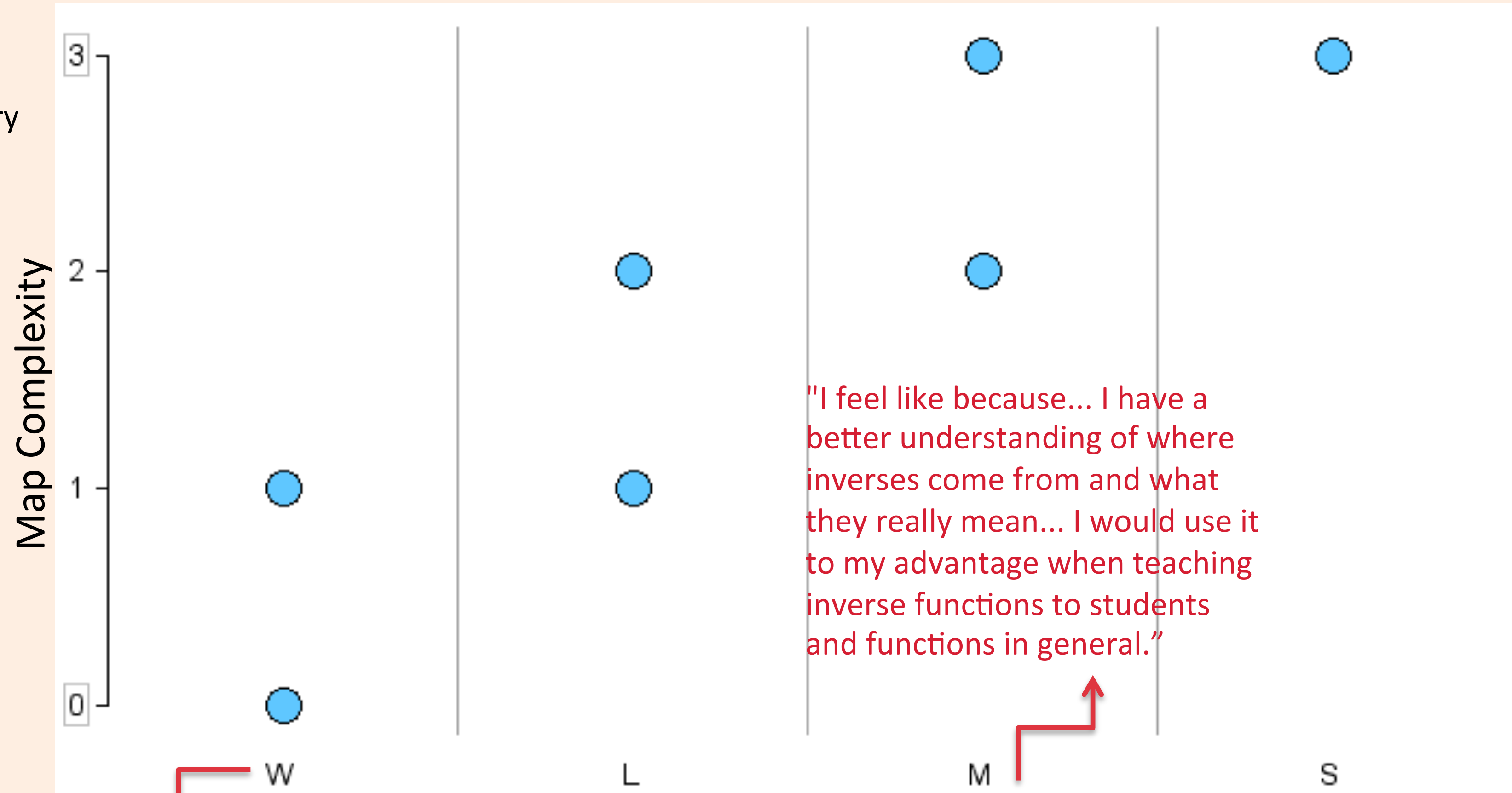
# Concept Maps & Course Performance

<u>*y-axis*</u>
0 = Simple
1 = Elementary
2 = Moderate
3 = Complex



# Concept Maps & Professional Value

y-axis
0 = Simple
1 = Elementary
2 = Moderate
3 = Complex




Impact Rating on Teaching

“I don’t see any relation”

x-axis
W = Weak
L = Low
M = Medium
S = Strong

# Final Thoughts

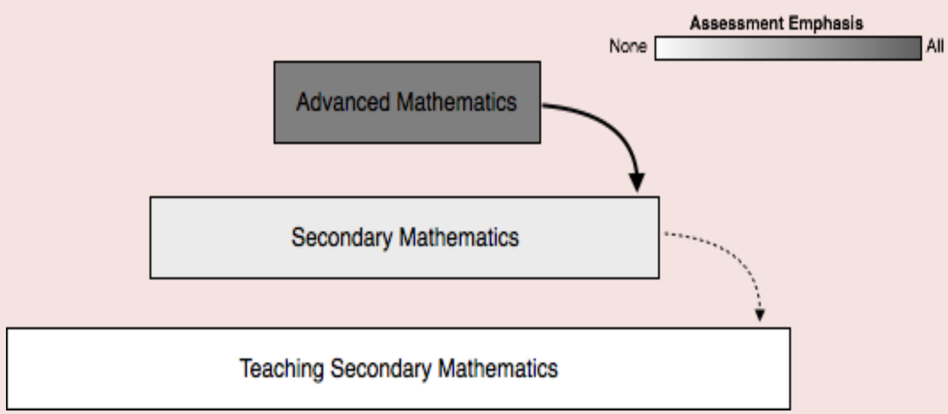
- High performance in an Abstract Algebra course did not guarantee that a teacher was making the desired connections between the mathematical concepts of the course and the content they teach – only smaller subset made these deep connections.
  - Making these connections explicit may be beneficial.
- The reported impactfulness of an Abstract Algebra course correlated strongly with map complexity – a task related to teaching not just course content. The results suggest that those making deeper connections with the material find more professional/teaching value in the course content.

# Real Analysis

- Real Analysis is often required for secondary teachers but teachers often find the course of little value (e.g., Goulding, Hatch, & Rodd, 2003; Zazkis & Leikin, 2010). Our interviews with teachers suggest little appreciation even for common things that mathematics educators deem important (0.999… = 1) – a need to reduce the transfer gap for teachers.
- Research Question
  – How might you design tasks in a real analysis course for teachers that is meaningful and related to teachers' professional needs and simultaneously faithful to advanced mathematics content?

# Real Analysis for Teachers

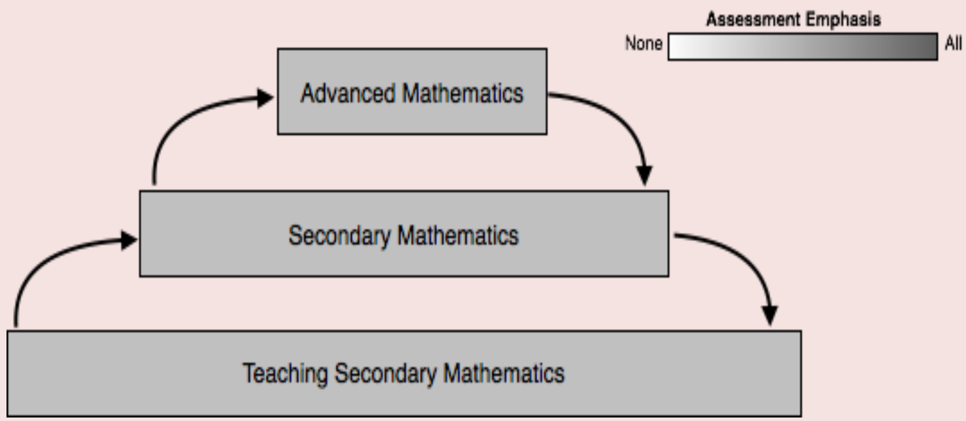
Traditional model




Trickle down effect: implicit hope is that a byproduct of learning advanced mathematics will be responding differently to instructional situations in the future.

# Real Analysis for Teachers

Modified model




"Building up from" and "stepping down to" practice.

General Argument:

i. Teachers must do X (some non-negotiable teaching practice)
ii. Part of doing X well is Y.
iii. The content of Real Analysis is well-situated to learn Y.

# Ex. 1: Solving Equations & Inverse Functions

- Teachers are responsible for providing feedback on students work and secondary teachers must teach students about the algebraic solving process (X); part of this involves navigating the terrain of inverse functions (Y). Real Analysis covers important notions connected to visualizing inverse functions.

# Ex. 1: Solving Equations & Inverse Functions

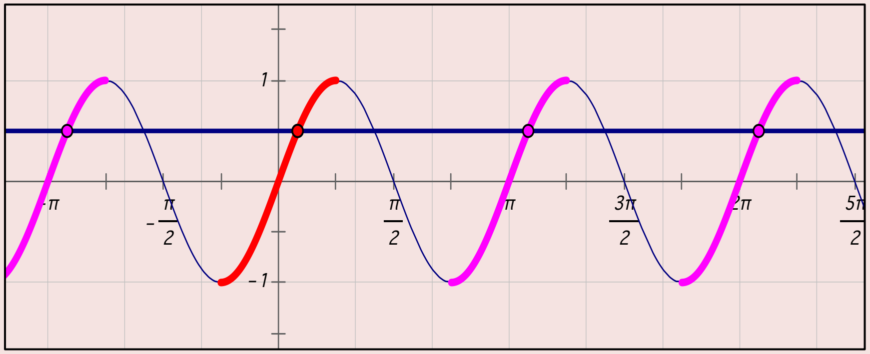
- **Task:** A student presents his/her work as follows.
  - How would you respond to the student and their work? What is the fundamental issue that arose in their work?
  - Give a statement and a rigorous argument about when an inverse function exists.
  - Describe how this notion can be visualized in the students' equation-solving.

$\sin(2x) = 0.5$

$2x = \arcsin(0.5)$ (take arcsin of both sides)

$2x = \pi / 6 + 2\pi k$ (evaluate arcsin(0.5), add period)

$x = \pi / 12 + \pi k$ (divide both sides by 2)



# Ex. 2: Scope of an argument

- Teachers are responsible for making content explicit through explanation, which often involves using analogies and descriptions to make sense of ideas (X) (e.g., exponents are repeated multiplication); doing so requires being aware of the extent to which an explanation or argument applies (Y) - oftentimes based on attention to *number sets*. Progression of ideas and proofs in real analysis often develop through number sets.

# Ex. 2: Scope of an argument

- **Task:**
  - Pick examples to use to teach students the power rules for derivatives.
  - Consider the proof: for what number sets (e.g., N,Z,Q,R) for *n* does it make sense – when does the argument break down?
  - Change the proof so that it extends to the next set of numbers.
  - For what sets/objects does the following make sense: "Exponents are repeated multiplication"

a. For some number $x_0$, the difference of powers formula states:

$$x^n - x_0^n = (x - x_0)\left(x^{n-1} + x^{n-2}x_0 + \ldots + x \cdot x_0^{n-2} + x_0^{n-1}\right) \text{ for all } x.$$

b. According to the definition, $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$. Thus:

$$f'(x_0) = \lim_{x \to x_0} \frac{x^n - x_0^n}{x - x_0} = \lim_{x \to x_0} \frac{(x - x_0)\left(x^{n-1} + x^{n-2}x_0 + \ldots + x \cdot x_0^{n-2} + x_0^{n-1}\right)}{x - x_0}$$

$$= \lim_{x \to x_0} x^{n-1} + x^{n-2}x_0 + \ldots + x \cdot x_0^{n-2} + x_0^{n-1}$$

c. There are *n* terms approaching ${x_0}^{n-1}$, so $f'(x_0) = nx_0^{n-1}$ for all $x_0$.

# Final Thoughts

- Teachers frequently do not develop desired connections nor do they place similar value on these mathematical connections (tension between rigor and relevance)
- Some potential considerations:
  - More closely align advanced mathematics with facets of *teaching* secondary mathematics
    - Instill mathematical practice of *considering the scope of an argument or claim*, which is relevant for teaching
    - Instill mathematical practice of *considering contingency assumptions*, which is relevant for teaching
  - Alongside rigorous proof, utilize more informal arguments such as graphical or technological approaches

# Participant Discussion

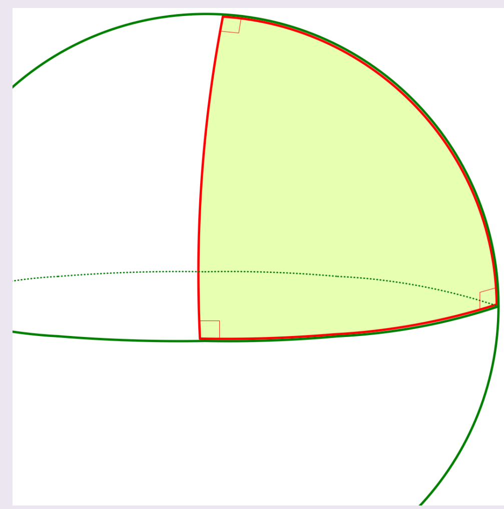
- What are some implications that you see from these three studies?
- What kinds of understandings about more advanced mathematics may be particularly productive or important for teaching based on these three studies?

# Forms of Knowing Advanced Mathematics for Teaching

- Many connections/bridges exist between more advanced mathematics and secondary content, some perhaps more meaningful for teaching. Analyzing broadly across examples and vignettes from CCSS-M analysis may inform particularly productive kinds of understandings.
- Research Question
  - In what way(s) should teachers know more advanced mathematics in order to foster understandings relevant for teaching of the given standard?

# Example Connection

- 8.G.A.5. Use informal arguments to establish facts about the angle sum…of triangles… *For example, arrange three copies of the same triangle so that the sum of the three angles appears to form a line…*
- HSG.CO.C.10. Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°...*



- Mr. Reese's use of Non-Euclidean geometry to engage students (via cognitive conflict) in critical thinking and proof about planar triangles (interior angle sum) – despite students familiarity with the 180º idea (Wasserman & Stockton, 2013)

# FOKs

- Forms of knowing advanced mathematics for teaching:
  - How simple things become complex later on **(Complexity)**
  - How mathematics systems are rooted in specific axiomatic foundations **(Axiomatic)**
  - How mathematical ideas evolve(d) **(Developed/ Evolutionary)**
  - How mathematical reasoning employs logical structures and valid rules of inference **(Logical)**
  - How statistical inference differs from other forms of mathematical reasoning **(Statistical/Inferential)**

# FOK1 Examples

- FOK1: How simple things become complex later on
  - Exponents: *CCSS-M 6.EE.1. Write and evaluate numerical expressions involving whole-number exponents*
    - Gets Complicated: Repeated multiplication models whole-number exponents, but breaks down with integer, rational, radical, and complex exponents encountered later on (e.g. CCSS-M 8.EE.1, HSN-RN.1)
  - Functions: 8.F.1-8.F.5, F.IF.1-F.IF.9, F.BF.1-F.BF.5
    - Gets Complicated: Assumed domain/range can vary (e.g. functions of complex instead of real numbers), as does the relationship between function analysis and graphs - polar instead of rectangular coordinates (e.g. vertical line test)
- Understanding that these notions get complicated reinforces the necessity to focus on their underlying meaning and not just tricks or shortcuts

# Final Thoughts

- Teachers' draw on/use their content knowledge as they teach; FOKs attempt to describe how teachers might understand more advanced mathematics in order to draw on it in their practice.
  - In mathematical explanation, teacher questioning, sequencing lessons, ability to *trim* mathematical ideas appropriate to student level relates to understanding more advanced ideas as "becoming complex"
- Teacher Educators might use these Forms of Knowing as a way to explore more advanced mathematics with their students, and in ways that connect to teachers' work and practices
  - For some specific K-12 mathematics content, asking "How does this mathematical idea get complicated later on?"
  - How might that influence your explanation, sequencing, or questioning around the content?

# Synthesis Across Studies

- Difficulty helping teachers acquire understanding of advanced mathematics as a key developmental understanding (abstract algebra, unbiased estimator) – speaks to need to identify better ways to teach such content (real analysis, FOKs)
- For those few teachers: i) found the content as more professionally relevant for their teaching; and ii) indication of influencing their approach teaching (unbiased estimator, abstract algebra) – speaks to need to further look into productivity/teaching benefit
- Teachers found more informal (less formal) arguments and approaches useful for their professional needs (unbiased estimator, real analysis) - speaks to need to utilize both while discussing more advanced ideas

# Synthesis & Participant Discussion

- What implications from this work might there be for mathematics teacher education practice and policy?
- How might some of the findings from these studies be incorporated into existing mathematics courses? Mathematics education courses?

# Conclusions

- Other thoughts? Questions?

Thanks!