Knowledge for teaching: Horizons and mathematical structures

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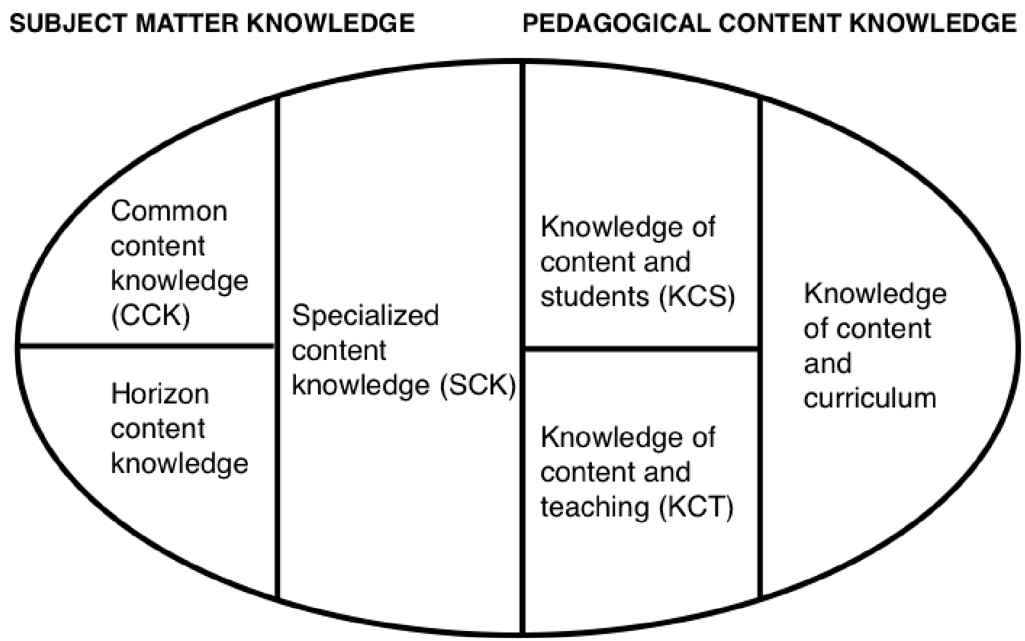
Context of the Study

- Overarching Question:

In what aspects of K-12 mathematics might advanced mathematics fundamentally change teachers' understandings about and perceptions of the content they teach – particularly in ways that might productively influence their (planned) teaching practices?

Introduction

- Mathematical Knowledge for Teaching (MKT), Ball, Thames, Phelps (2008)



Knowledge at the Mathematical Horizon

Ball and Bass (2009) identify four components:

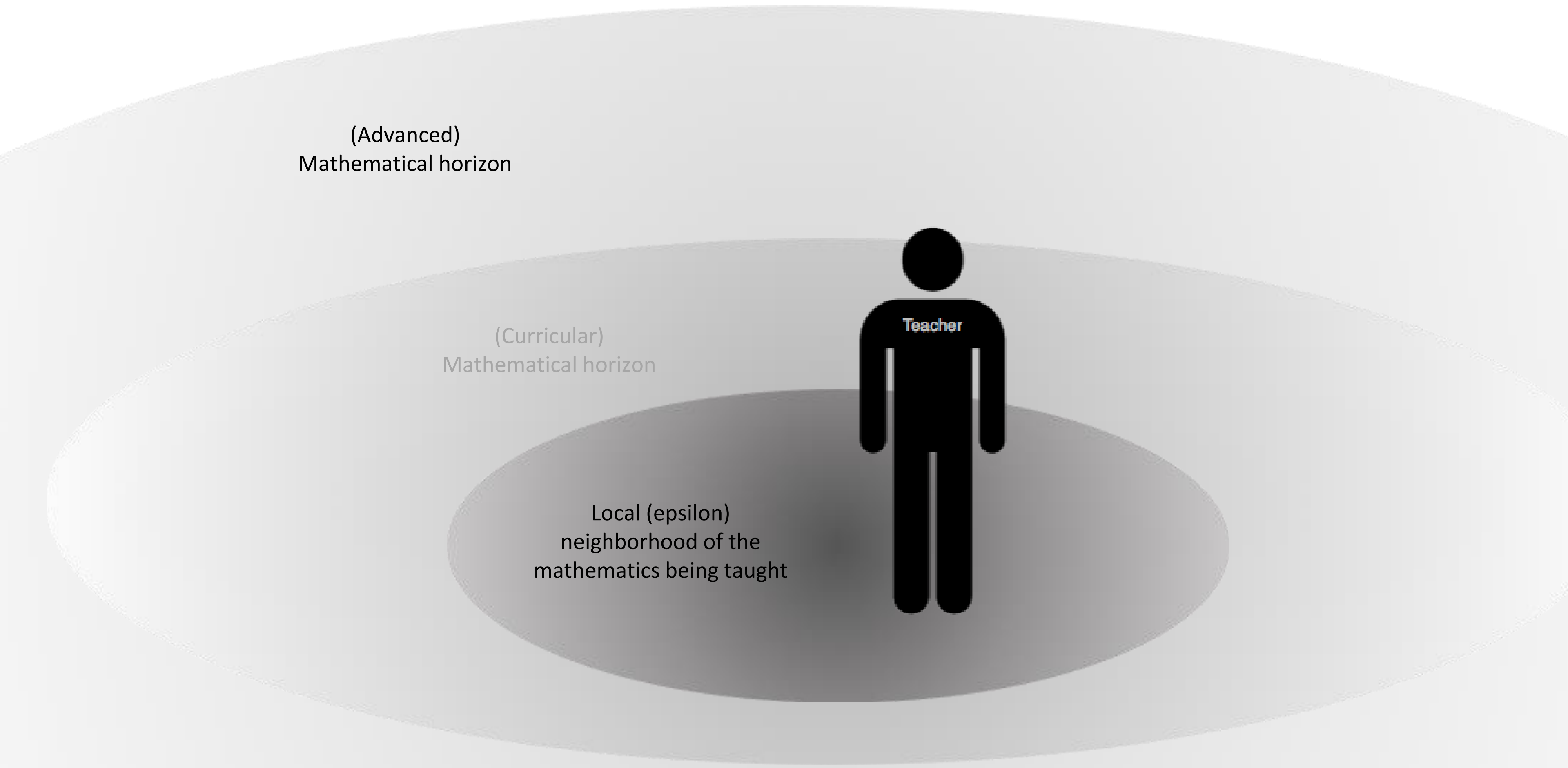
1. A sense of the mathematical environment surrounding the current "location" in instruction (larger mathematical landscape)

2. Major disciplinary ideas and structures

3. Key mathematical practices

4. Core mathematical values and sensibilities

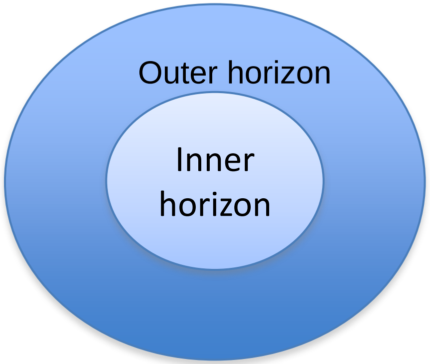
Mathematical Landscape



HCK Perspectives

- Transformational perspective
  - HCK as characterized in part *by transformation of teachers' own understandings* about and perceptions of the content they teach – in the sense that the *content is seen in a new light*, that the meaning or *understanding of ideas is shifted*, or that the *content is re-organized, re-ordered, or re-structured* in the teachers' mind.

"KMH" Perspectives



Adapting Husserl's "horizons"

- Inner Horizon: corresponds to aspects of an object that are not at the focus of attention but that are also intended

- Outer Horizon: includes features which are not in themselves aspects of the object, but which are connected to the world in which the object exists (Follesdal, 1998, 2003).

Can be applied to analyse what is in and out of focus for individuals and what connections they make

Specific Research Questions:

- What mathematical structures underpin and connect the content areas across elementary, middle, and secondary school mathematics, and how can these structures provide avenues for transforming individuals' perceptions of said content? **In particular, what are the transformative possibilities of algebraic structures on more elementary content?**

- **In what ways can awareness of a mathematical object's periphery provide opportunities for connecting mathematics content to their underlying structures?**

Findings

- Inverse Functions
  - What goes through your mind when considering the following statement?

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x$$

  - Understood in the group structure – set of invertible functions under composition
  - Action-Process-Object-Schema (APOS)

Findings

- Inverse Progression
  - 5.NF.B.7a. Interpret division of a unit fraction by a non-zero whole number, and compute such quotients. *For example, create a story context for (1/3) ÷ 4, and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that (1/3) ÷ 4 = 1/12 because (1/12) × 4 = 1/3.*
  - 7.NS.A.2b. Understand that integers can be divided, provided that the divisor is not zero, and every quotient of integers (with non-zero divisor) is a rational number. If p and q are integers, then $-(p/q) = (-p)/q = p/(-q)$. Interpret quotients of rational numbers by describing real-world contexts.
  - HSN.VM.C.10. (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.
  - HSF.BF.B.4. Find inverse functions.
    - 4a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. For example, $f(x)=2x^3$ or $f(x)=(x+1)/(x-1)$ for $x \neq 1$.
    - 4b. (+) Verify by composition that one function is the inverse of another.

Findings

- Influence on teaching
  - More coherent development of idea across multiple instantiations of it; $i(x)=x$ gains broader sense of importance; cancellation process – inverse/identity connection; teachers own development along APOS
  - During an introduction to inverse trigonometric functions, for example, a secondary teacher may ask students: *Compare the following: –sin(x), csc(x), $sin^{-1}(x)$. Are any the same? Are they all different? Rewrite –sin(x) and csc(x) using a "-1" somewhere; why do all of these use "-1"?*
    - Notably, $\sin x+(-\sin x)=0$, $(\sin x)(\csc x)=1$, and $\sin(\sin^{-1}(x))=x$, which also reiterates the connection of an inverse to an identity element.

Conclusions

Content Area	Elementary School	Middle School	High School
Arithmetic Properties	Properties of Addition, Multiplication on N, Q+	Properties of Addition, Multiplication on Z, Q, R; Algebraic expressions	Properties of Addition, Multiplication on R, C; Polynomials, Rationals Properties of Matrix multiplication; Properties of Function composition
	Fields / Rings	⟷	Groups
	Concrete	⟷	Abstract
Inverses	Inverse Operations	Transitioning to Inverse Elements	Inverse Elements and Inverse Functions
	→	→	Comprehensive Framework for Inverses
Structure of Sets	Equivalence within number sets (identity)	Equivalence within algebraic expressions (arithmetic properties); Expansion on number sets to Z, Q (closure)	Expansion to C (closure); Parallel structures across number sets and algebraic expressions
	Within Sets	⟷	Across Sets
Solving Equations	(Guess and check)	Systematic solving (arithmetic operations)	Systematic solving (complex functions); Systems of equations
	Arithmetic Operations	⟷	Complex Functions

Conjectures

- Teacher education: Beginning with K-12 content areas might provide a rationale/motivation for studying more advanced structures, connecting them not just to the K-12 content but also to the teaching of that content within its progression across school mathematics

- Undergraduate education: Structure, advanced mathematics, and connected understandings have broad applicability and explicit attention to these in undergraduate education might have wide reaching implications for student retention and engagement

Participant Discussion

- In what ways could knowledge of mathematical structure be useful for school teaching?
- How could developing explicit links amongst school content and its underlying structure form part of prospective teachers’ mathematical preparation?
- What experiences at the undergraduate level could be relevant for developing teachers’ horizon knowledge, with a focus on transforming perceptions of elementary content?