A rationale for irrationals

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History

- Mathematicians from many early cultures assumed that all positive numbers could be written as a ratio of two natural numbers (i.e., rational numbers)
  - Infinitely many rational numbers between any two numbers (no matter how close) seemed to "cover" all possibilities 
  - Archimedes, for example, found π to be between $^{223}/_{71}$ and $^{22}/_{7}$ (based on $\sqrt{3}$ being between $^{1351}/_{780}$ and $^{265}/_{153}$)

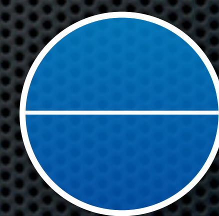
Irrational Numbers

- Irrational numbers, however, are the solution to many interesting (and simple) problems in mathematics:
    - The diagonal of a unit square

    - The circumference of a unit circle

π

- For various reasons, π is frequently students' first introduction to irrational numbers
  - Circles are common
  - Derives from linear/length measurements



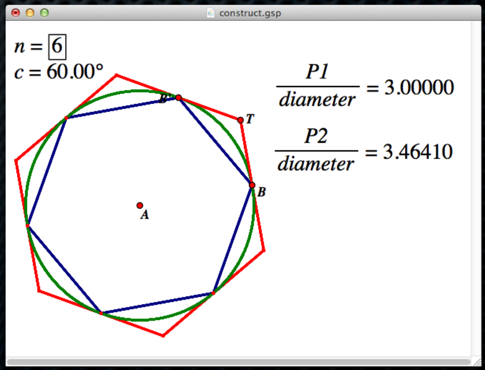
- In 7th grade (CCSS-M), students are expected to know the formulas for circumference and area of a circle and use them to solve problems (7.G.B.4)

Irrational Numbers

- In 8th grade (CCSS-M), students are expected to be knowledgeable about the *behavior of irrational numbers* (8.NS.A.1, 8.NS.A.2) [including Pythagorean Theorem, 8.G.B.6-8; roots 8.EE.A.2]
- Yet a common tendency in classrooms and on standardized tests **is to avoid irrational (and rational) solutions to problems in favor of integers**
  - Easier for students to check solutions, to grasp, etc.
- Without practice, students' appreciation of their importance and their (infinite) behavior may be limited

Irrational Numbers

- So how can we help students understand the behavior of irrational numbers and appreciate their importance?
  - Iterations of squared numbers that get closer to 13 (Lewis, 2007)
  - Regular polygons and dynamic software to study the ratio between circumference and diameter (Wasserman & Arkan, 2011)
- **In this session, a task that led to an unexpected exploration of another irrational number is presented**



Overview

- First, we will discuss the original task (and review the 8 Standards for Mathematical Practice)
- Second, we will discuss “Problem Posing” (Brown & Walter, 2005), which led to an unintended exploration by modifying the task
- Last, we will discuss the modified task (and some extensions)

Standards for Mathematical Practice

- **MP.1.** Make sense of problems and persevere in solving them.
- **MP.2.** Reason abstractly and quantitatively.
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- **MP.4.** Model with mathematics.
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- **MP.6.** Attend to precision.
- **MP.7.** Look for and make use of structure.
- **MP.8.** Look for and express regularity in repeated reasoning.

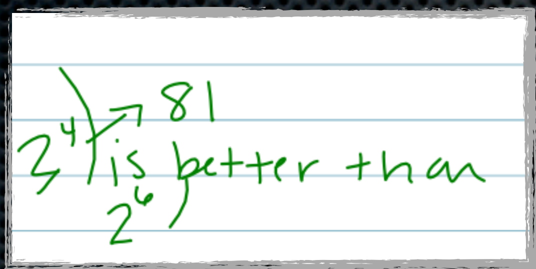
Original Task

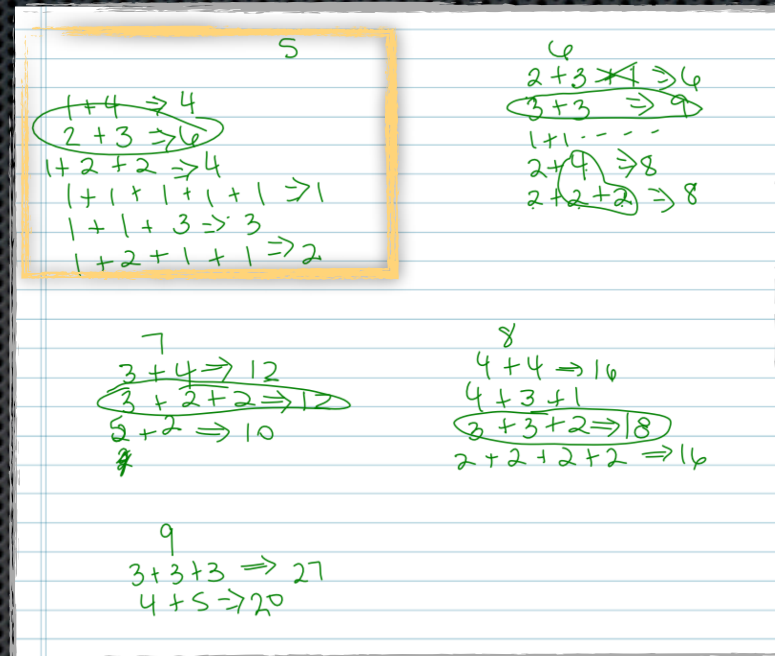
Find positive integers whose sum is 2012 and whose product is as large as possible.

(In other words, what is the positive integer partition of 2012 that results in a maximum product?)

Solution Method 1

- Solve a simpler problem





- Splitting into 3s provides a larger product because $2+2+2+2+2+2=3+3+3+3$ but $2^6<3^4$. (Also, $2^3<3^2$.)
- 2012 splits into 335 *sixes* with a remainder of 2: therefore, maximum product is: **3^{670} * 2**

Solution Method 2

- Reasoning through Experimentation

$2012 \div 4 = 503$

$503^4 = 64{,}013{,}554{,}081$

$503 + 503 + 503 + 503$

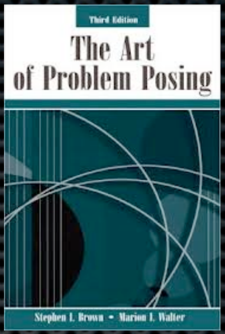
$4 + 4 + 4 \ldots$ 503 times

4^{503}

"commutative"

- *"The more amounts we have to multiply by, the more numbers there are in the sum, the larger the [product]."*
  - Some groups, then, falsely concluded that the largest exponent would make the largest product, i.e., 2^{1006}
- This led to an intriguing question: **Why is splitting into groups of 3 better than groups of 2 in this problem?**

Problem Posing



- Brown & Walter's (2005) book, *The Art of Problem Posing*, refers to shifting the mathematical involvement of students to include **not just the solving of problems but also their formation**.
- Various strategies for "problem posing" are discussed; one, in particular, is the "What-If-Not" Approach:
  - List specific attributes of a problem and then explore implications for changing or removing one or more of them
  - Example: Pythagorean Theorem, suppose *not equal* ($a^2+b^2<c^2$), or *not addition* ($a^2 - b^2=c^2$)

A Modified Task

Find positive *numbers* whose sum is 2012 and whose product is as large as possible.

(In other words, what is the real number partition of 2012 that results in a maximum product?)

NOTE: the word “integers” was replaced with “numbers”

Solution Method 2

- From the "commutative" observation of Solution method 2, dividing 2012 into groups of equal size, the product in consideration was: $4^{2012/4}$, $503^{2012/503}$, $6^{2012/6}$
- Generalization and algebraic manipulation led to:

  $2012 \div a = \frac{2012}{a}$

  $(a)^{\frac{2012}{a}} \longrightarrow \left(a^{\frac{1}{a}}\right)^{2012}$

- Maximizing the interior function, $f(x) = x^{1/x}$, leads to a maximum product

Solution Method 2

- Whether you use iteration (or guess and check) to find the maximum product...

$2^{1/2} = 1.4142$

$3^{1/3} = 1.44224$

$4^{1/4} = 1.414$

$10^{1/10} = 1.2589$

$2.5^{1/2.5} = 1.44269$

$2.6^{1/2.6} = 1.444126$

$2.9^{1/2.9} = 1.44360$

$2.7^{1/2.7} = 1.444655$

$\rightarrow 2.8^{1/2.8} = 1.4444$

$2.75^{1/2.75} = 1.444632$

$2.71^{1/2.71} = \cancel{1.51236}\ 1.444665$

$2.7111^{1/2.7111} = 1.444665$

2.719

$2.7109^{1/2.7109} = 1.444665$

Solution Method 2

- Or you use Calculus...

$$y = a^{\frac{1}{a}}$$

$$\ln y = \ln a^{1/a}$$

$$\ln y = \frac{1}{a}\ln a$$

$$\frac{1}{y}y' = \left(\frac{1}{a}\right)\left(\frac{1}{a}\right) + \left(\ln a\left(-\frac{1}{a^2}\right)\right)$$

$$y' = \frac{y}{a^2} + y\ln a\left(-\frac{1}{a^2}\right)$$

$$y' = \frac{a^{1/a}}{a^2} + a^{1/a}\ln a\left(-\frac{1}{a^2}\right)$$

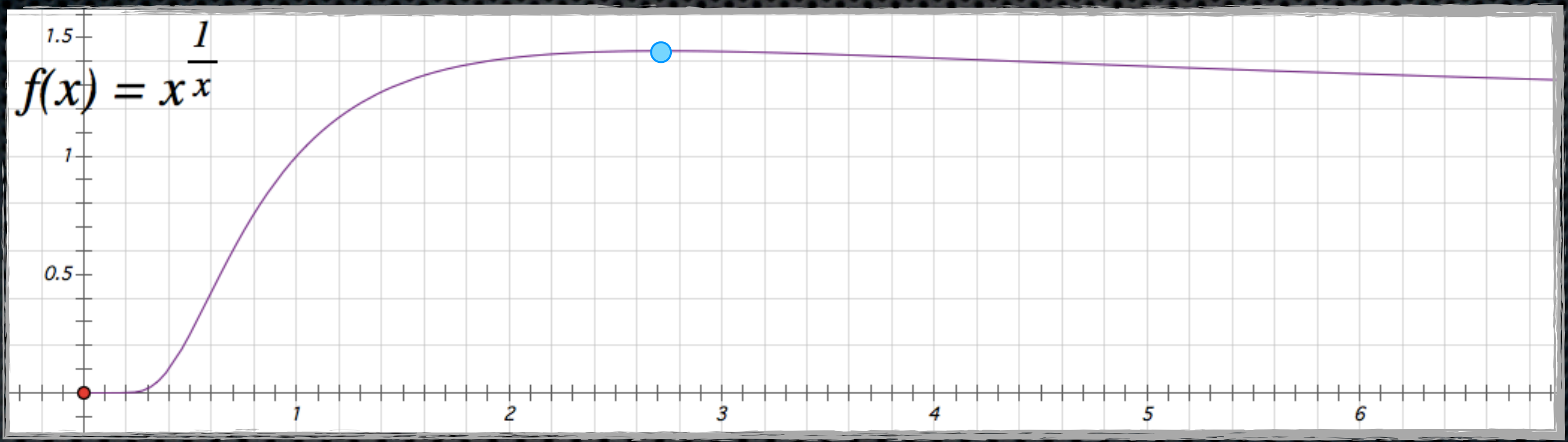
$$0 = a^{1/a}/a^2\,(1 - \ln a)$$

$$\ln a = 1$$

$$a = e$$

Solution Method 2

- Or you look at a graph...



e

- All explore the **irrational number *e* as the solution to this problem** - *the partition of a number that results in a maximum product*
- While iteration (guess and check) cannot *prove* that the solution is **irrational** (as opposed to **rational**), it does provide an opportunity to discuss the *behavior of irrational numbers,* especially compared to rational numbers.
  - Infinite decimal expansion of rational numbers repeat
  - NOTE: with 10 digits in the decimal (2.7182818284), the continued fraction form shows an infinite pattern, making the solution irrational

$$2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{8+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\ }}}}}}}}}}}}}}$$

Solution Method 1

- You can employ similar reasoning as Solution Method 1 to arrive at the same conclusion, e.g., $12=2+2+2+2+2+2 = 3+3+3+3$ but $2^6<3^4$.
- For any composite number, expressed as the product of two factors, x and y, which is bigger: x^y or y^x ?
- Assuming one is smaller, $x^y < y^x$, leads to: $\frac{\ln x}{x} < \frac{\ln y}{y}$
- The maximum of this function, $f(x) = \frac{\ln x}{x}$, is also e.

Caveat

- **A precise solution...**
  - Theoretically the optimal solution, maximum product, would be: $e^{2012/e}$, or $\sim 2.718...^{740.173...}$
  - However, the *irrational exponent* does not traditionally represent a product...
- So the precise solution requires finding a rational approximation for *e*: either **$^{2012}/_{740}$** (~2.7189...) or **$^{2012}/_{741}$** (~2.71525...)
  - The precise solution is $\left(2012/740\right)^{740}$

Interesting Extension

- While the exponents are too large for a calculator to compute, students could use *properties of logarithms* to compare various products on a calculator.
  - For example, is $3^{670} * 2$ bigger than 2^{1006} ?
  - Yes, since: $670 * \log 3 + \log 2 > 1006 * \log 2$

Introductions to *e*

- The irrational number *e* is frequently discussed in the context of **continuously compounded interest**, which results from the fact that

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e$$

- Or in Calculus as the exponential function with the property that: ***f'(x) = f(x)***
- The modified task relied only on knowledge of basic operations, which would allow introduction to the irrational number *e* *before limits* are required.

Conclusion

- The goal, however, is not to introduce students to a specific irrational number, *but to understand the nature of irrational numbers and appreciate their importance*.
- In this regard, the modified task could provide students with an opportunity to:
  - **Engage in important mathematical practices**

Standards for Mathematical Practice

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- In this regard, the modified task could provide students with an opportunity to:
  - **Engage in important mathematical practices**
    - MP.6. Attend to precision: e.g., rational approximations of irrational numbers
  - **Discuss the infinite behavior of irrational numbers**
  - **Connect irrational numbers as the solution to real problems**

Thank you!

Questions? Comments?

(NOTE: look for upcoming article in MT)

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